Lecture 30

Mechanical waves.

Transverse waves.

Sound waves.

What is a wave ?

A wave is a traveling disturbance that transports energy but not matter.

Examples:
- Sound waves (air moves back & forth)
- Stadium waves (people move up & down)
- Water waves (water moves up & down)
- Light waves (what moves??)

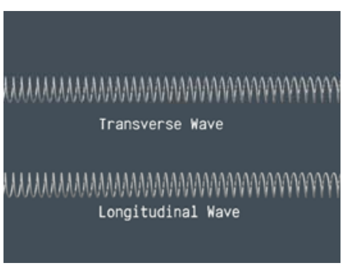
Waves exist as excitations of a (more or less) elastic medium.

Types of Waves

DEMO:
Rope, slinky and wave machines

Transverse: The medium oscillates perpendicular to the direction the wave is moving.

- String
- Water

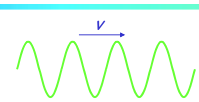


Longitudinal: The medium oscillates in the same direction the wave is moving

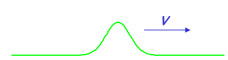
- Sound
- Slinky

Forms of waves

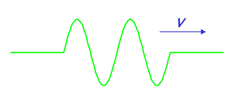
- Continuous or periodic: go on forever in one direction
  - → in particular, harmonic (sin or cos)



- Pulses: brief disturbance in the medium

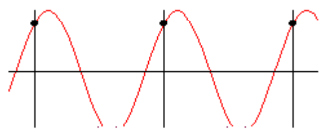


- Pulse trains, which are somewhere in between.



Harmonic waves

Each point has SHM



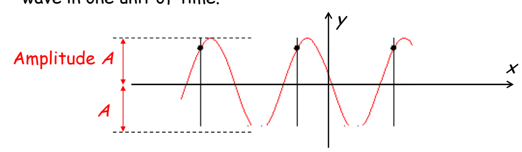
A few parameters

Amplitude: The maximum displacement A of a point on the wave.

Period: The time T for a point on the wave to undergo one complete oscillation.

Frequency: Number of oscillations f for a point on the wave in one unit of time. $f = \frac{1}{T}$

Angular frequency: radians ω for a point on the wave in one unit of time. $\omega = 2\pi f$

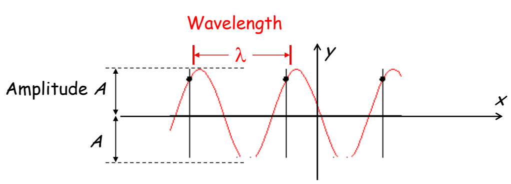


Connecting all these SHM

Wavelength: The distance λ between identical points on the wave.

Speed: The wave moves one wavelength λ in one period T, so its speed is

$$v = \frac{\lambda}{T} = \lambda f$$



Wave speed

The speed of a wave is a constant that depends only on the medium:

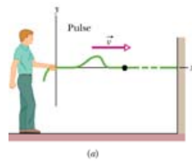
→ How easy is it to displace points from equilibrium position?
→ How strong is the restoring force back to equilibrium?

Speed does NOT depend on amplitude, wavelength, period or shape of wave.

ACT: Waves on a string

A single pulse is sent along a stretched rope. What can the person do to make *the start of the pulse* arrive at the wall in a shorter time?

A. Flick hand faster
B. Flick hand further up and down
C. Pull on rope before flicking hand



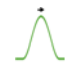
Wave speed in a string

$$v = \sqrt{\frac{F}{\mu}}$$

(see appendix 1)

Pulling on rope increases tension, and propagation speed depends only on medium, not on how you start the wave.

Faster flick up/down → narrow pulses



Slower flick up/down → wider pulses

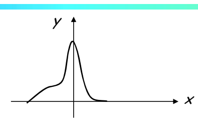
Large flick up/down → higher pulses

Once pulse leaves your hand, you cannot influence it.

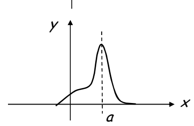
Propagation speed down string is ~ same for all these pulses

Mathematical description of a wave

Suppose we have some function $y = f(x)$:

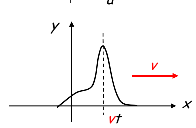


$f(x - a)$ is just the same shape moved a distance a to the right:



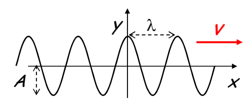
Let $a = vt$

Then, $f(x - vt)$ will describe the same shape moving to the right with speed v.



Math for the harmonic wave

Consider a wave that is harmonic in x and has a wavelength λ:



If $y = A$ at $x = 0$:

$$y(x) = A\cos\left(\frac{2\pi}{\lambda}x\right)$$

If this is moving to the right with speed v:

$$y(x,t) = A\cos\left(\frac{2\pi}{\lambda}(x - vt)\right)$$

Different forms of the same thing

$$y(x,t) = A\cos\left(\frac{2\pi}{\lambda}(x - vt)\right)$$

We knew: $v = \frac{\lambda}{T} = \lambda\frac{\omega}{2\pi}$

Define: $k = \frac{2\pi}{\lambda}$ Wave number

$$y(x,t) = A\cos(kx - \omega t)$$

$$v = \frac{\omega}{k}$$

ACT: Wave Motion

A harmonic wave moving in the positive x direction can be described by the equation $y(x,t) = A\cos(kx - \omega t)$

Which of the following equations describes a harmonic wave moving in the negative x direction?

(a) $y(x,t) = A\cos(kx + \omega t)$

(b) $y(x,t) = A\cos(-kx + \omega t)$

(c) Both

$y(x,t) = A\cos(kx - \omega t)$ came from $y(x,t) = A\cos\left(\frac{2\pi}{\lambda}(x - vt)\right)$

$y(x,t) = A\cos(kx + \omega t) \quad \leftrightarrow \quad -x$ direction

$\cos(-kx + \omega t) = \cos(kx - \omega t) \quad \leftrightarrow \quad +x$ direction

It's the relative sign that matters.

The wave equation

General wave: $y = f(x - vt)$ Let $u = x - vt$

$$\frac{\partial y}{\partial t} = -v\frac{\partial f}{\partial u} \qquad \frac{\partial y}{\partial x} = \frac{\partial f}{\partial u}$$

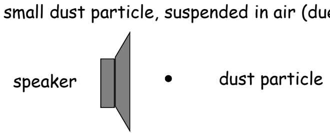
$$\frac{\partial^2 y}{\partial t^2} = v^2\frac{\partial^2 f}{\partial u^2} \qquad \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 f}{\partial u^2}$$

$$\frac{\partial^2 y}{\partial t^2} = v^2\frac{\partial^2 y}{\partial x^2}$$

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = 0$$ Wave equation

ACT: Dust in front of loudspeaker

Consider a small dust particle, suspended in air (due to buoyancy)



When you turn on the speaker, the dust particle

A. oscillates back and forth horizontally, and moves slowly to the right
B. steadily moves to the right
C. oscillates back and forth horizontally

DEMO: Candle

Pressure/density oscillations

Gas in equilibrium: pressure and density are uniform.

Sound wave: periodic longitudinal oscillations of particles in the gas

Consider one slice of air:

1. Oscillation to the right causes pressure to increase
2. Increase in force causes neighboring air to be displaced
   → sound wave propagates
3. Slice of air oscillates back to region of low pressure

The small volumes of air do not propagate with wave, they only oscillate around their equilibrium position.

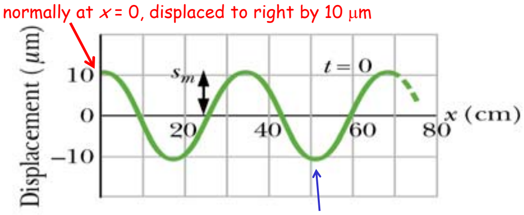
DEMO: Sound in vacuum.

Harmonic longitudinal waves

Consider a gas in long, thin, horizontal tube. Each particle of gas oscillates horizontally in a harmonic way:

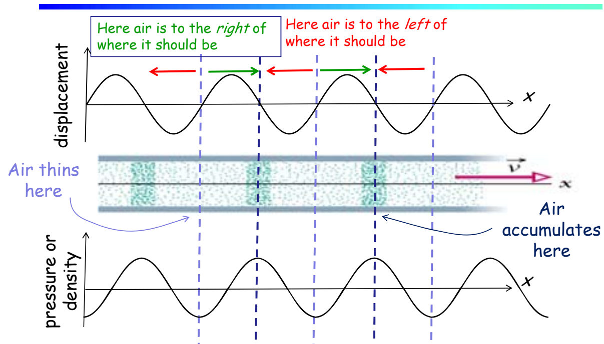
$$s(x,t) = s_{max}\cos(kx - \omega t)$$

air normally at $x = 0$, displaced to right by 10 μm



air normally at $x = 50$ cm, displaced to left by 10 μm

Pressure, density oscillations



Zero displacement ↔ Maximum density and pressure

Displacement waves and density/pressure waves are out of phase by 90°.

$$s(x,t) = s_{max}\cos(kx - \omega t)$$

$$\Delta p(x,t) = \Delta p_{max}\sin(kx - \omega t)$$
$$\Delta \rho(x,t) = \Delta \rho_{max}\sin(kx - \omega t)$$

Wave speed

DEMO:
Organ pipe with
different gases.

In general: $v = \sqrt{\dfrac{\text{restoring force property}}{\text{inertial property}}}$

String: $v = \sqrt{\dfrac{F}{\mu}}$

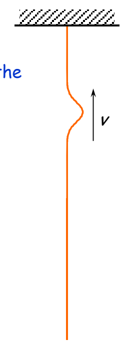
Sound in a fluid: $v = \sqrt{\dfrac{B}{\rho}}$ (see appendix 3)

Sound in a solid: $v = \sqrt{\dfrac{Y}{\rho}}$

Mythbuster's clip

ACT: Wave Motion

A heavy rope hangs from the ceiling, and a small amplitude transverse wave is started by jiggling the rope at the bottom. As the wave travels up the rope, its speed will:



(a) increase

(b) decrease

(c) stay the same

Tension is greater near the top because it has to support the weight of rope under it!

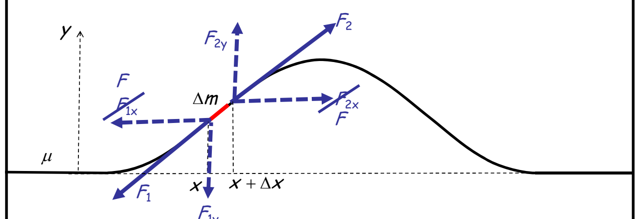
$v = \sqrt{\dfrac{F}{\mu}}$ $\Rightarrow$ v is greater near the top.

Appendix 1: Wave speed

Problem: A pulse travels in the $+x$ direction in a string with mass per unit length of the string is μ (kg/m) subject to a uniform tension F.

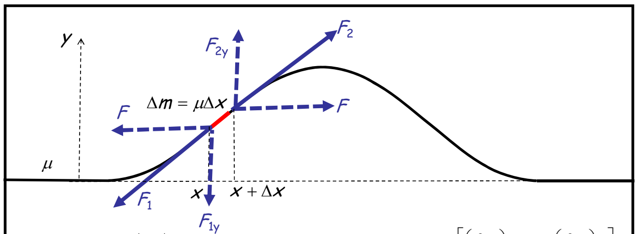
What is the speed of the pulse?

Consider the segment of length Δx when the string is relaxed: $\Delta m = \mu \Delta x$



$|F_{1x}| = |F_{2x}|$ because $a_x = 0$ (transversal wave, no displacement in the x direction)

F_x must also be equal to the tension in the string when there is no wave, ie, $|F_{1x}| = |F_{2x}| = F$



At x: $\dfrac{|F_{1y}|}{F} = -\left(\dfrac{\partial y}{\partial x}\right)_x$

At $x+\Delta x$: $\dfrac{|F_{2y}|}{F} = \left(\dfrac{\partial y}{\partial x}\right)_{x+\Delta x}$

$$F_{y\ \text{net}} = |F_{2y}| - |F_{1y}| = F\left[\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_x\right] = \Delta m\ a_y = \mu \Delta x\ \frac{\partial^2 y}{\partial t^2}$$

$$F\left[\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_x\right] = \mu \Delta x\ \frac{\partial^2 y}{\partial t^2}$$

$$F\left[\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_x\right] = \mu \Delta x\ \frac{\partial^2 y}{\partial t^2}$$

$$\frac{\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_x}{\Delta x} = \frac{\mu}{F}\ \frac{\partial^2 y}{\partial t^2}$$

$$\lim_{\Delta x \to 0} \frac{\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_x}{\Delta x} = \frac{\partial^2 y}{\partial x^2}$$

Wave equation! $\dfrac{\partial^2 y}{\partial x^2} = \boxed{\dfrac{\mu}{F}} \dfrac{\partial^2 y}{\partial t^2}$, with $\dfrac{\mu}{F} \rightarrow \dfrac{1}{v^2}$

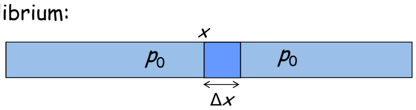
Wave speed in a string

$$\boxed{v = \sqrt{\frac{F}{\mu}}}$$

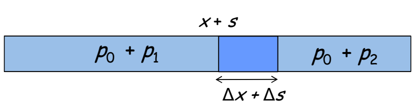
Appendix 2: Relation between displacement and pressure

Consider a pipe of cross-sectional area A filled with air, and a small element at x with thickness Δx.

In equilibrium:



Due to a wave, element moves and changes its size



Pressure and displacement are related through the bulk modulus of the air!

$$B = -\frac{p}{\frac{\Delta V}{V}}$$

← This is the *gauge* pressure (actual pressure minus equilibrium pressure)

$V = A\Delta x$

$\Delta V = A\Delta s$

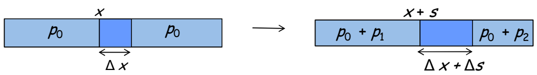
$$\frac{\Delta V}{V} = \frac{\Delta s}{\Delta x} \xrightarrow{\Delta x \to 0} \frac{\partial s}{\partial x}$$

$$\boxed{p(x,t) = -B\frac{\partial s(x,t)}{\partial x}}$$

The harmonic case: Out of phase as predicted

$$s(x,t) = s_{max}\cos(kx - \omega t) \quad \rightarrow \quad p(x,t) = \underbrace{Bks_{max}}_{p_{max}}\sin(kx - \omega t)$$

Appendix 3: Sound wave speed



Net force on the element: $F = (p_1 - p_2)A$

Acceleration of the element: $a = \dfrac{\partial^2 s}{\partial t^2}$

Mass of the element: $\Delta m = \rho A \Delta x$

$$(p_1 - p_2)A = \rho A \Delta x \frac{\partial^2 s}{\partial t^2}$$

$$\frac{(p_1 - p_2)}{\Delta x} = \rho \frac{\partial^2 s}{\partial t^2} \xrightarrow{\Delta x \to 0} -\frac{\partial p}{\partial x} = \rho \frac{\partial^2 s}{\partial t^2}$$

$$p = -B \frac{\partial s}{\partial x}$$

$$\frac{\partial p}{\partial x} = -B \frac{\partial^2 s}{\partial x^2}$$

$$-\frac{\partial p}{\partial x} = \rho \frac{\partial^2 s}{\partial t^2}$$

$$\rho \frac{\partial^2 s}{\partial t^2} = B \frac{\partial^2 s}{\partial x^2}$$

$$\frac{\partial^2 s}{\partial x^2} - \frac{\rho}{B} \frac{\partial^2 s}{\partial t^2} = 0$$

Wave equation with $v = \sqrt{\dfrac{B}{\rho}}$

Pressure and density oscillations

It all boils down to a phase difference:

Displacement $s(x,t) = s_{max} \cos(kx - \omega t)$

Pressure $p(x,t) = p_{max} \sin(kx - \omega t)$

Note that *p is the gauge pressure*. The pressure of air in equilibrium is p_{atm}. The oscillations give a total pressure $p_{total}(x,t) = p_{atm} + p(x,t)$

Density $\Delta\rho(x,t) = \Delta\rho_{max} \sin(kx - \omega t)$

Density oscillations are also about the regular air density. Total density is $\rho_{total}(x,t) = \rho_0 + \Delta\rho(x,t)$

ACT: Frequency and wavelength

The speed of sound in air is a bit over 300 m/s, and the speed of light in air is about 300,000,000 (3×10^8) m/s.

Suppose we make a sound wave and a light wave with a wavelength of 3 m each.

What is the ratio of the frequency of the light wave to that of the sound wave?

- (a) About 10^6
- (b) About 10^{-6}
- (c) About 1000

$$f = \frac{v}{\lambda} \qquad \frac{v_{light}}{v_{sound}} \sim 10^6 \qquad \frac{f_{light}}{f_{sound}} \sim 10^6$$

What are these frequencies???

For sound having λ = 3 m : $f = \frac{v}{\lambda} \sim \frac{300 \text{ m/s}}{3 \text{ m}} = 100 \text{ Hz}$ (bass hum)

For light having λ = 3 m : $f = \frac{v}{\lambda} \sim \frac{3\times10^8 \text{ m/s}}{3 \text{ m}} = 100 \text{ MHz}$ (FM radio)