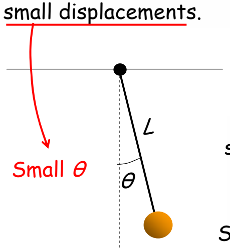
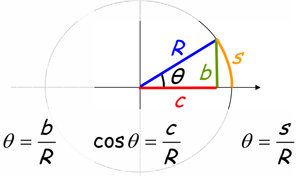
Lecture 29

The Pendulum
Damped and Forced Oscillations

The simple pendulum

A mass *m is* suspended at the end of a massless string of length *L*. Find the frequency of the oscillations for small displacements.





$\sin\theta = \frac{b}{R}$ $\cos\theta = \frac{c}{R}$ $\theta = \frac{s}{R}$

Small θ → $b \simeq s$, $c \simeq R$ → $\boxed{\sin\theta \approx \theta \quad \cos\theta \approx 1}$

The oscillations are rotations about point P.

$$\tau_{net} = \tau_g = mgL\sin\theta$$

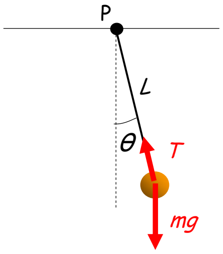
Newton's second law: $(\tau_{net} = I\alpha)$

$$-mgL\sin\theta = mL^2\frac{d^2\theta}{dt^2}$$

Small θ

$$-mgL\theta = mL^2\frac{d^2\theta}{dt^2}$$

$$\boxed{\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0}$$ SHM



DEMO: Pendulum

Solution:

$$\theta = \theta_0\cos(\omega t + \varphi)$$

with $\boxed{\omega = \sqrt{\frac{g}{L}}}$

The physical pendulum

= a rigid body that oscillates about an axis (P).

Oscillations about P: $(\tau_{net} = I\alpha)$

$$-mgd\sin\theta = I\frac{d^2\theta}{dt^2} \longrightarrow \frac{d^2\theta}{dt^2} + \frac{mgd}{I}\sin\theta = 0$$

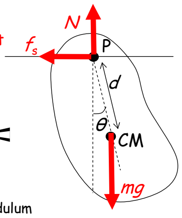
Distance between CM and suspension point

Moment of inertia about suspension point

For small θ: $\frac{d^2\theta}{dt^2} + \frac{mgd}{I}\theta = 0$

$$\boxed{\text{SHM with } \omega^2 = \frac{mgd}{I}}$$

DEMO: Physical pendulum

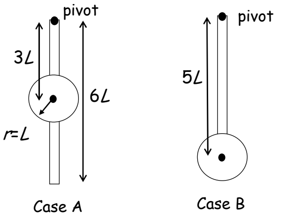


Example: Rod and disk

A rod of length $6L$ long has a disk with $r = L$ attached to it. The disk can be positioned along the rod's length. Both the rod and the disk have mass m. The distance between the center of the disk and the pivot point is $3L$ in case A and $5L$ in case B.

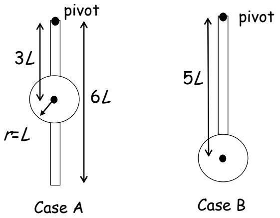
Find the ratio ω_A/ω_B.

- A. 1.31
- B. 1.14
- C. 1.00
- D. 0.874
- E. 0.764



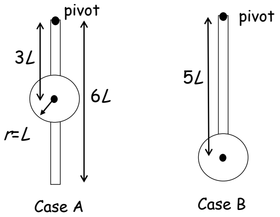
Angular frequency of a physical pendulum:

$$\omega = \sqrt{\frac{mgd}{I}}$$



$$\frac{\omega_A}{\omega_B} = \frac{\sqrt{\dfrac{(2m)gd_A}{I_A}}}{\sqrt{\dfrac{(2m)gd_B}{I_B}}} = \sqrt{\frac{d_A I_B}{d_B I_A}}$$

$$\frac{\omega_A}{\omega_B} = \sqrt{\frac{d_A I_B}{d_B I_A}}$$



A: $d_A = 3L$

$$I_A = \frac{1}{3}m(6L)^2 + \frac{1}{2}mL^2 + m(3L)^2 = 21.5mL^2$$

B: $d_B = \dfrac{m5L + m3L}{2m} = 4L$

$$I_B = \frac{1}{3}m(6L)^2 + \frac{1}{2}mL^2 + m(5L)^2 = 37.5mL^2$$

$$\frac{\omega_A}{\omega_B} == \sqrt{\frac{d_A I_B}{d_B I_A}} = \sqrt{\frac{3L \times 37.5mL^2}{4L \times 21.5mL^2}} = \sqrt{\frac{112.5}{86}} = 1.14$$ Answer B

$$\frac{T_A}{T_B} = \frac{\omega_B}{\omega_A} = \frac{1}{1.14} = 0.874$$

Lowering the disk increases the period (ie, slows the pendulum down).

This is how you tune a grandfather clock.

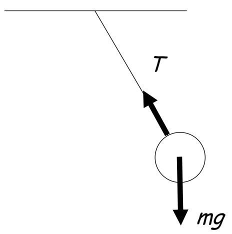
DEMO:
Metronome

ACT: Swing

A person swings on a swing. When the person sits still, the swing moves back and forth at its natural frequency. If, instead, the person stands on the swing, the new natural frequency of the swing is:

- [✓ A. Greater]
- B. The same
- C. Smaller

$\omega = \sqrt{\dfrac{g}{L}}$ But where did that come from?



$\tau = I\alpha$ with $\tau = Lmg\sin\theta$

L is really the distance from the axis of rotation to the CM (the point where the weight vector is applied).

If the person stands, L becomes smaller.

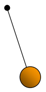
⇩

ω increases.

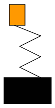
Damped oscillations

Life is not ideal…

Pendulums don't swing forever. Some friction or damping force slows it to a stop.



Pendulum in air or other fluid



Spring with mass and drag (car suspension)

Linear damping (by a fluid, for small speeds): the damping force is proportional to velocity.

$$\vec{F}_D = -b\vec{v}$$

Damped Harmonic Motion

Example: Mass and spring

Newton's 2nd law: $F_{net} = -kx - bv = ma$

$$\downarrow$$

$$-kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

$$\downarrow$$

$$\frac{d^2x}{dt^2} + \left(\frac{b}{m}\right)\frac{dx}{dt} + \left(\frac{k}{m}\right)x = 0 \quad \text{Diff. eqn.}$$

Three types of solutions:

$b < 2\sqrt{km}$ underdamping

$b = 2\sqrt{km}$ critical damping

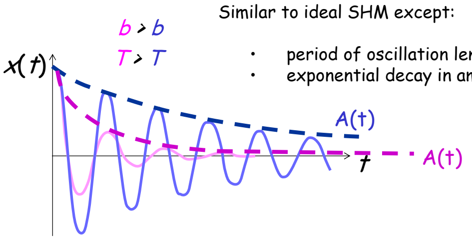
$b > 2\sqrt{km}$ overdamping

Underdamping $\left(b < 2\sqrt{km}\right)$

$\boxed{x = A(t)\cos(\omega' t + \varphi)}$ with $\boxed{A(t) = Ae^{-(b/2m)t} \quad \omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}}$

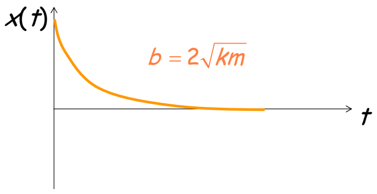
Similar to ideal SHM except:

- period of oscillation lengthened
- exponential decay in amplitude



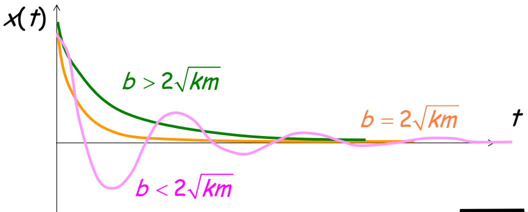
Critical damping $\left(b = 2\sqrt{km}\right)$

$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} = 0!!$ No oscillations!



Overdamping $\left(b > 2\sqrt{km}\right)$

No oscillations either. The decay is slower than with critical damping.



VIDEO:
Sand pendulum

Forced Oscillations and Resonance

Damping is always present $\longrightarrow$ To keep a system going, we need to apply a driving force.

For a driving force of the form $F_{applied} = F_{\max}\cos(\omega_D t)$

the amplitude goes as: $A = \frac{F_{\max}}{\sqrt{(k - m\omega_D^2)^2 + b^2\omega_D^2}}$

Resonance

When $\boxed{\omega_D = \sqrt{\frac{k}{m}} = \omega_{natural}}$ A is very large, even for small $F_{\max}$

(without damping, ie $b = 0$, $A \to \infty$)

DEMO:
Resonance

VIDEO:
Galloping Gertie