Lecture 26

Simple Harmonic Motion

Periodic motion

A motion is called periodic when the system comes back to the same situation every time interval T.

- Same situation: Same position, Same velocity
- T: Period

Frequency $f = \frac{1}{T}$

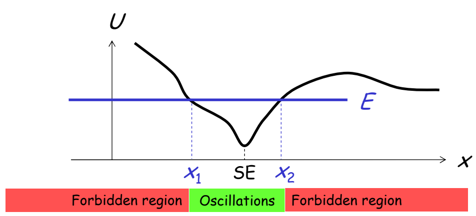
Angular frequency $\omega = 2\pi f = \frac{2\pi}{T}$

Examples:

- Uniform circular motion
- Earth around the Sun
- Toy train in a circuit

Oscillations about an equilibrium position

When a 1D system is released near a stable equilibrium point, the motion is periodic: oscillations between two turn-around points x_1 and x_2.



Simple Harmonic Motion (SHM)

SHM is the oscillatory motion that happens when the restoring force is proportional to the displacement from the equilibrium position:

$$F \propto -x$$

↕ equivalent to

...or when the potential energy is a quadratic function of the displacement from the equilibrium position:

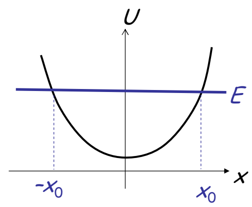
$$U \propto x^2 \text{ (a parabola)}$$

Example of SHM: Spring that obeys Hooke's law

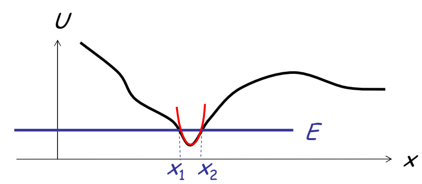
$F = -kx$

Oscillations about $x = 0$

$U = \frac{1}{2}kx^2$



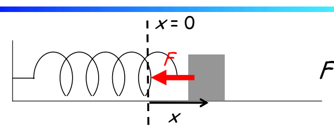
Most oscillatory systems can be approximated by SHM when the oscillations are small enough:



If not too far from the minimum, the curve is approximately a parabola.

Technically: Taylor's expansion of $U(x)$ up to the quadratic term

The SHM equation



$F = -kx$

Newton's second law for the block: $-kx = ma$

$$m\frac{d^2x}{dt^2} + kx = 0$$

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$ SHM eqn.

with $\omega = \sqrt{k/m}$ (spring)

Solutions to this differential equation: $x = \sin\omega t$ $\quad x = \cos\omega t$

General solution (standard form): $x = A\cos(\omega t + \phi)$

SHM equation: $\frac{d^2x}{dt^2} + \omega^2 x = 0$

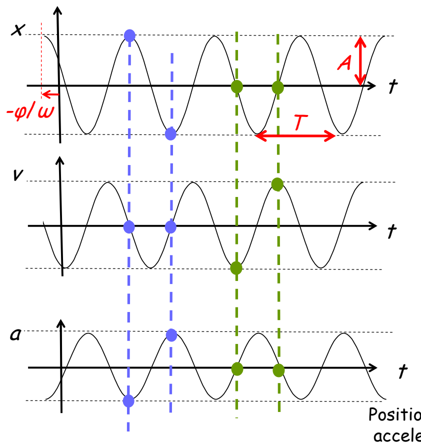
General solution: $x = A\cos(\omega t + \phi)$

- A: Amplitude
- ω: Angular frequency
- ϕ: Phase angle or phase

$x = A\cos(\omega t + \phi)$ $\qquad |x_{MAX}| = A$

$v = \frac{dx}{dt} = -A\omega\sin(\omega t + \phi)$ $\qquad |v_{MAX}| = A\omega$

$a = \frac{dv}{dt} = -A\omega^2\cos(\omega t + \phi) = -\omega^2 x$ $\qquad |a_{MAX}| = A\omega^2$



Max $|x|=A$

$v=0$

Max $|a|=A\omega^2$

$x=0$

Max $|v|=A\omega$

$a=0$

DEMO:
Position, velocity and acceleration in SHM

ACT: Amplitude

DEMO:
Amplitude and frequency

Two identical masses hang from two identical springs. In case 1, the mass is pulled down 2 cm and released. In case 2, the mass is pulled down 4 cm and released. How do the periods of their motions compare?

A. $T_1 < T_2$
B. $T_1 = T_2$
C. $T_1 > T_2$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{k}{m}}} = 2\pi\sqrt{\frac{m}{k}}$$

Does not depend on the amplitude!

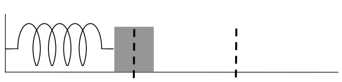
This is general to all SHM (not only for springs):

Period is independent of amplitude.

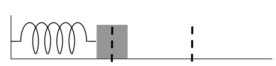
Example: SHM spring

A 3-kg block is attached to the end of a spring with k = 1000 N/m. The spring is compressed 25 cm as shown and released at t = 0. What is the velocity of the mass 10 s later? (Use the given x-axis).

A. 4.57 m/s
B. -4.57 m/s
C. 1.73 m/s
D. -1.73 m/s
E. 0.00 m/s



A 3-kg block is attached to the end of a spring with k = 1000 N/m. The spring is compressed 25 cm and released at t = 0. What is the velocity of the mass 10 s later?



We need to determine the $x(t)$ function.

$$x(t) = A\cos(\omega t + \varphi) \qquad v(t) = -A\omega\sin(\omega t + \varphi)$$

- A = 25 cm = 0.25 m
- $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000\text{ N/m}}{3\text{ kg}}} = 18.26\text{ s}^{-1}$
- $\begin{cases} -0.25 = 0.25\cos\varphi \\ 0 = -A\omega\sin\varphi \end{cases}$ $\begin{cases} \cos\varphi = -1 \\ \sin\varphi = 0 \end{cases}$ $\varphi = \pm\pi$

$$v(t) = -A\omega\sin(\omega t + \varphi) = -4.57\sin(18.26t + \pi)\text{ m/s}$$

$$v(10\text{ s}) = +1.73\text{ m/s}$$

Answer C

4.57 m/s (Answers A and B) is the maximum speed

ACT: Initial conditions

Let's consider the same example with the horizontal spring, but now we set the clock to zero when the block is passing through the equilibrium position with velocity in the positive-x direction.

If $x = A\cos(\omega t + \varphi)$, determine the phase φ.

A. $\varphi = 0$ B. $\varphi = +\pi/2$ **C. $\varphi = -\pi/2$**

$x = A\cos(\omega t+\varphi)$ → $0 = A\cos\varphi$ → $\cos\varphi = 0$

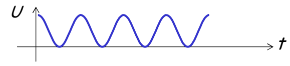
$v = -A\omega\sin(\omega t+\varphi)$ → $A\omega = -A\omega\sin\varphi$ → $\sin\varphi = -1$

→ $\varphi = -\pi/2$

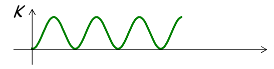
Energy in SHM

The SHM driving force is conservative, so mechanical energy is always conserved. The energy oscillates back and forth between the kinetic and potential forms.

$$U(t) = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2(\omega t)$$



$$K(t) = \frac{1}{2}mv^2 = \frac{1}{2}mA^2\omega^2\sin^2(\omega t)$$

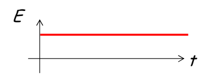


$$E = KE(t) + U(t) = \frac{1}{2}mA^2\omega^2\sin^2(\omega t) + \frac{1}{2}kA^2\cos^2(\omega t)$$

$$= \frac{1}{2}kA^2\left[\sin^2(\omega t) + \cos^2(\omega t)\right]$$

$$= \frac{1}{2}kA^2 \quad \text{(constant!)}$$

$$mA^2\omega^2 = mA^2\frac{k}{m} = kA^2$$



ACT: Energy oscillations

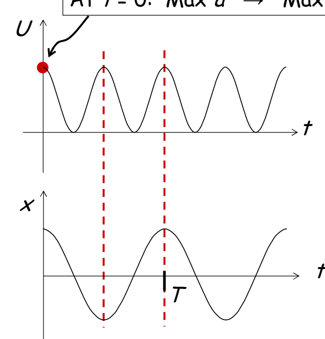
A mass $m = 5$ kg oscillates at the end of a spring of constant $k = 2000$ N/m. At $t = 0$, its acceleration is maximum. How long will it take before the potential energy reaches its next maximum?

A. 0.31 s

B. 0.16 s

C. 0.08 s

At $t = 0$: Max a → Max x → Max U



To get to the next peak in U, it takes **half** a period

$$\frac{T}{2} = \frac{1}{2}\frac{2\pi}{\omega} = \pi\sqrt{\frac{m}{k}} = 0.16 \text{ s}$$