Lecture 27

Newton's Law of Gravity

Newton and the Moon

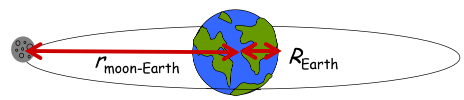
What is the centripetal acceleration of the Moon for its rotation around the Earth?

We know (Newton also knew):

T = 27.3 days = 2.36×10^6 s

$r_{\text{moon-Earth}} = 3.84 \times 10^8$ m

$R_{\text{Earth}} = 6.35 \times 10^6$ m (see Appendix)



Angular frequency of the moon: $\omega = \dfrac{1 \text{ rotation}}{27.3 \text{ day}} = \ldots = 2.66 \times 10^{-6} \text{s}^{-1}$

$a_c = r\omega^2 = 0.00272 \text{ m/s}^2 = 0.000278\, g$ pointing toward the Earth

$$\frac{a_c}{g} = 0.000278$$

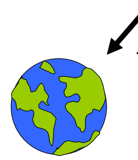
Newton noticed that

$$\frac{R_E^2}{r^2} = 0.000273$$

This inspired him to suggest that $F_{\text{gravity}} \propto 1/r^2$

$$a = \frac{\text{constant}}{r^2}$$

$$g = \frac{\text{constant}}{R_E^2}$$



Newton's Law of Gravitation

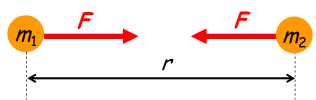
Two point-like bodies of mass m_1 and m_2 that are separated by a distance r attract each other with a force with magnitude

$$F = G\frac{m_1 m_2}{r^2}$$

$$G = 6.67 \times 10^{-11}\ \frac{\text{Nm}^2}{\text{kg}^2}$$

Gravitational constant

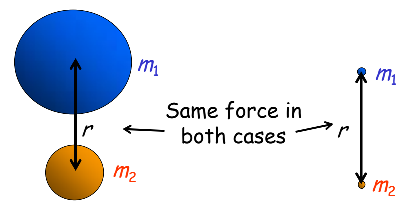
and direction along the line between both bodies.



Spherical shell theorem

An important result that we will not prove until Gauss's law in 222.

The gravitational force exerted by a spherically symmetric object of radius R and mass M is the same as the force exerted by a particle of mass M located at the center of the sphere, for distances $r \geq R$.



Weight

Example of application of the spherical shell theorem:

Object near the surface of the Earth

Force by the Earth on an object of mass m at distance h from the ground

$$F_g = G\frac{mM_E}{(R_E + h)^2} \simeq G\frac{mM_E}{R_E^2} = mg$$

$R_E \gg h$

$$g \equiv G\frac{M_E}{R_E^2}$$

ACT: Earth model

Paul's weight on Earth is 1000 N. What will be his weight if he stands on the surface of a scale model of the Earth, made of the same type of material but scaled to half the size?

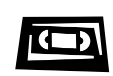
A. 250 N

B. 500 N

C. 4000 N

$$g_X = G\frac{M_X}{R_X^2} = G\frac{\frac{M_E}{2^3}}{\left(\frac{R_E}{2}\right)^2} = \frac{1}{2} g_E$$

Mass scales like volume



VIDEO: On the Moon

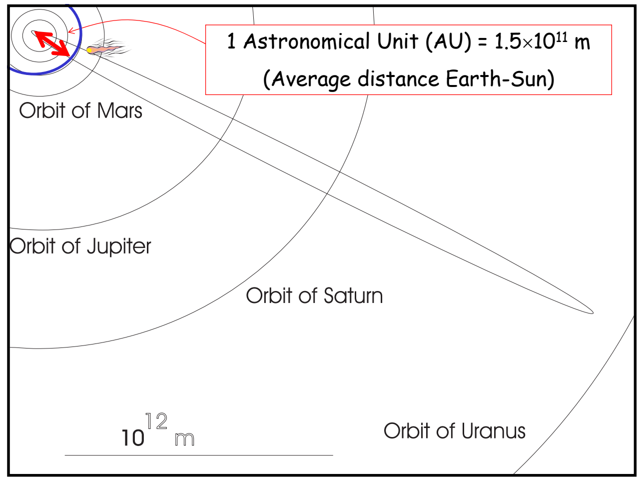
Gravity is a very weak force

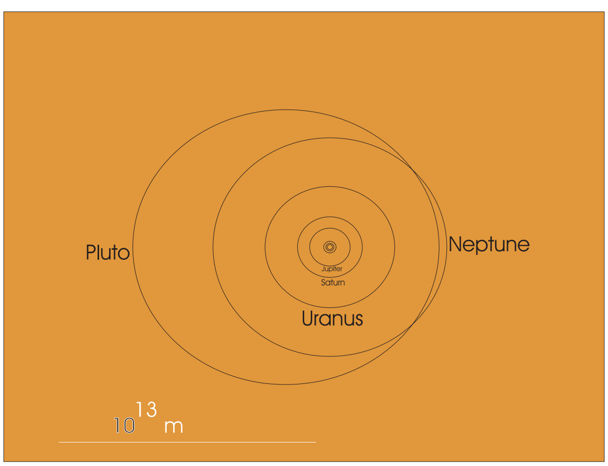
Example: Find the magnitude of the gravitational attraction between two 3-kg books sitting 1 cm apart.

$$F = (6.67\times10^{-11}\ \frac{\text{Nm}^2}{\text{kg}^2})\frac{(3\ \text{kg})^2}{(10^{-2}\ \text{m})^2} = 6.0\times10^{-6}\ \text{N}\quad !!!$$

Completely negligible.

Gravitational force matters when at least one of the objects is very massive, like stars, planets or galaxies. In fact, it is the dominant force in the structures at astronomical scale.

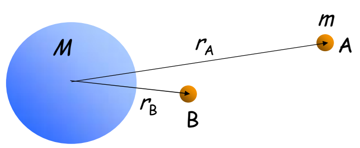




Gravitational potential

Gravitational forces are conservative.

Find the work done by gravity on an object of mass *m* that moves from point A to point B in the vicinity of a planet of mass *M*.



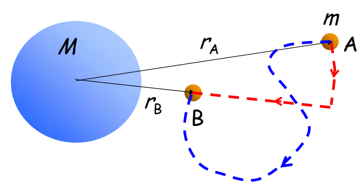
$$W_{A\to B} = \int \vec{F}\cdot d\vec{l} = -\int_{r_A}^{r_B} G\frac{Mm}{r^2}dr = -GMm\left[-\frac{1}{r}\right]_{r_A}^{r_B} = -\left(-\frac{GMm}{r_B}+\frac{GMm}{r_A}\right)$$

$$= -(U_B - U_A)$$

For any path, the dot product selects the radial part of $d\vec{l}$

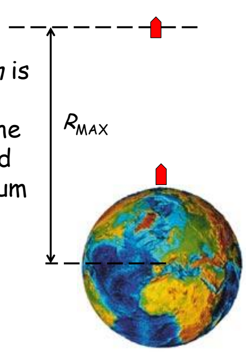
Gravitational potential energy

$$U_g(r) = -\frac{GMm}{r} + \text{constant}$$



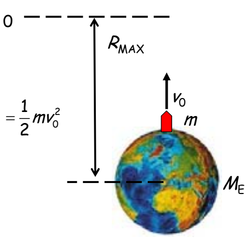
Example: How high?

A projectile of mass *m* is launched straight up from the surface of the Earth with initial speed v_0. What is the maximum distance from the center of the Earth it reaches before falling back down?



$$U_f = -G\frac{M_E m}{R_{MAX}} \qquad KE_f = 0$$

$$U_0 = -G\frac{M_E m}{R_E} \qquad KE_0 = \frac{1}{2}mv_0^2$$



Conservation of mechanical energy

$$-G\frac{M_E m}{R_E} + \frac{1}{2}mv_0^2 = -G\frac{M_E m}{R_{MAX}}$$

$$-\frac{1}{R_E} + \frac{1}{2}\frac{v_0^2}{GM_E} = -\frac{1}{R_{MAX}}$$

$$R_{MAX} = \frac{R_E}{1-\frac{v_0^2 R_E}{2GM_E}}$$

Note: Independent of *m*!

Example: Escape speed

What is the minimum initial speed of a projectile of mass m to escape from Earth?

A. 9.8 m/s
B. 9.8 km/s
C. 11.2 km/s
D. 46.4 km/s
E. 98 km/s

Escape = reach infinity (where the attraction of Earth is zero)

$$R_{MAX} = \frac{R_E}{1 - \frac{v_{esc}^2 R_E}{2GM_E}} \to \infty \quad \Leftrightarrow \quad 1 - \frac{v_{esc}^2 R_E}{2GM_E} = 0$$

$$v_{esc}^2 = \frac{2GM_E}{R_E} = 2gR_E$$

$$v_{esc} = \sqrt{2gR_E} \; (= 11.2 \text{ km/s})$$ Escape speed

Also independent of m, of course!

Types of orbits

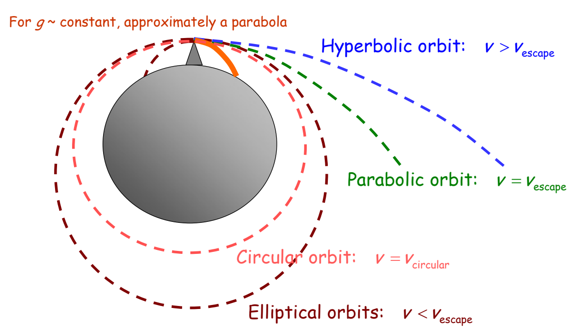
- Closed orbits (ellipses): $v < v_{escape} = \sqrt{2G\frac{M}{r}}$

  Special ellipse: Circular orbit $v_{circular} = \sqrt{G\frac{M}{r}}$

- Open orbits (parabolas and hyperbolas): $v \geq v_{escape}$

For the Earth, $v_{escape} = 11.2$ km/s

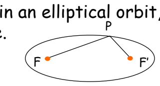
$v_{circular} = 7.9$ km/s



Kepler's Laws

Originally formulated for the planets in the Solar system, but they apply to any closed orbit (changing "Sun" by whatever applies).

1. Each planet moves in an elliptical orbit, with the Sun at one focus of the ellipse.

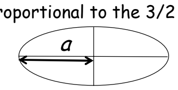


PF + PF' = constant

2. A line from the Sun to the given planet sweeps out equal areas in equal times
   $\Leftrightarrow$ Angular momentum about the Sun is conserved.

   $\Rightarrow$ Faster near the Sun

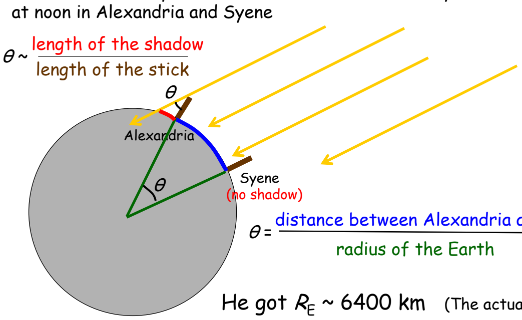
3. The periods of the planets are proportional to the 3/2 power of the semi-major axis a.



Appendix:
Measuring the radius of the Earth in 200 BC

Eratosthenes of Cyrene measured the shadow cast by a vertical stick at noon in Alexandria and Syene

$$\theta \sim \frac{\text{length of the shadow}}{\text{length of the stick}}$$



$$\theta = \frac{\text{distance between Alexandria and Syene}}{\text{radius of the Earth}}$$

He got $R_E \sim 6400$ km (The actual value!!!)