Lecture 24

Angular Momentum

Angular Momentum

We want an equivalent of $\vec{p}$ and of $\vec{F}_{net} = \frac{d\vec{p}}{dt}$

Angular momentum: $\vec{L} = \vec{r} \times \vec{p} = m(\vec{r} \times \vec{v})$

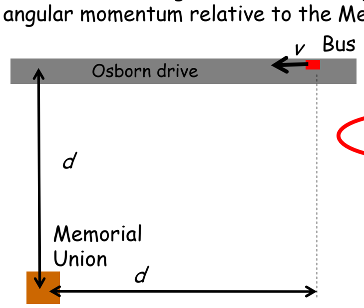
Just as linear momentum quantifies the amount of motion, angular momentum quantifies the amount of rotation.

Note that as with torque, the value depends on the center used.

Units: kg m^2/s = N m s = J s

ACT: Angular momentum

A bus of a mass m drives along Osborn Drive at speed v as shown in the figure below. The magnitude of its angular momentum relative to the Memorial Union is.



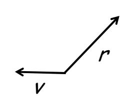
A. mvd

B. $\sqrt{2}mvd$

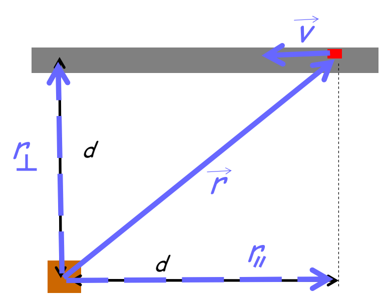
C. $2mvd$

$\vec{L} = \vec{r} \times \vec{p} = m\vec{r} \times \vec{v}$

Direction: into the sky (RHR)



Magnitude: $mvr_{\perp} = mvd$



Angular momentum for rigid bodies

Consider a rigid body being rotated around an axis perpendicular to the page and through point P. The body can be broken down into n particles.

$$\vec{L} = \sum_i \vec{L}_i = \sum_i \vec{r}_i \times \vec{p}_i = \sum_i \vec{r}_i \times m_i \vec{v}_i = \sum_i m_i r_i v_i \text{, out of the page}$$

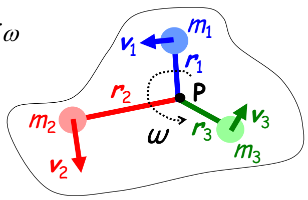
(if rotations about axis of symmetry)

Rigid structure: $v_i = r_i \omega$

$$\sum_i m_i r_i v_i = \left(\sum_i m_i r_i^2 \right) \omega = I \omega$$

$$\boxed{L_z = I \omega_z}$$

Angular momentum for rigid bodies
(for rotations about axis of symmetry)



Relation between angular momentum and torque

Let us look at the simplest case (one particle):

$$\frac{d\vec{L}}{dt} = \frac{d(\vec{r} \times \vec{p})}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}$$

$$= \vec{v} \times \vec{p} + \vec{r} \times \vec{F}_{\text{net}} = \vec{\tau}_{\text{net}}$$

$(\vec{v} \times \vec{p} = \vec{v} \times m\vec{v} = 0)$

$$\boxed{\frac{d\vec{L}}{dt} = \vec{\tau}_{\text{net}}}$$

equivalent of $\frac{d\vec{p}}{dt} = \vec{F}_{\text{net}}$

Conservation of angular momentum

The (usually) most interesting situation:

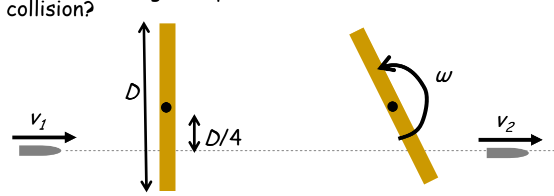
When the <u>net torque on a system is zero</u>, the total angular momentum of the system is conserved.

$$\frac{d\vec{L}}{dt} = 0 \qquad \vec{L}_{\text{initial}} = \vec{L}_{\text{final}}$$

EXAMPLE: Bullet hitting a stick

A uniform stick of mass M and length D is pivoted at the center. A bullet of mass m is shot through the stick at a point halfway between the pivot and the end. The initial speed of the bullet is v_1 and its final speed is v_2.

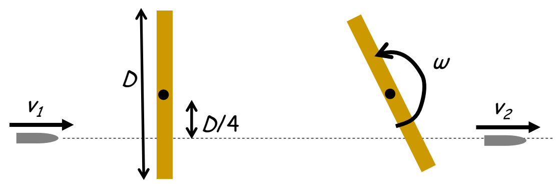
What is the angular speed ω of the stick after the collision?



External forces here:

- Weight of the stick → No torque about the axle
- Force on the stick by the pivoting axle → No torque about the axle
- Weight of the bullet → Negligible

No external torque ⟶ Angular momentum conserved



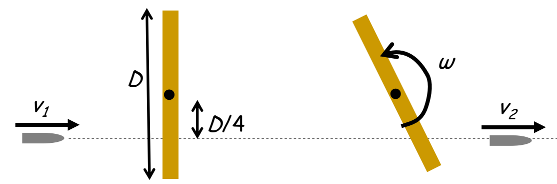
$$L_{z\,\text{before}} = mv_1\frac{D}{4}$$

$$L_{z\,\text{after}} = mv_2\frac{D}{4} + I\omega_z$$

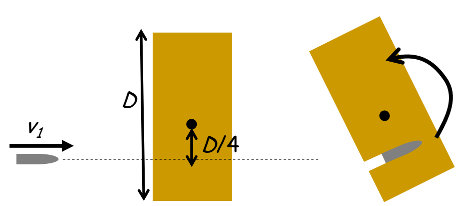
$$mv_1\frac{D}{4} = mv_2\frac{D}{4} + I\omega_z$$

$$\omega_z = \frac{mD(v_2 - v_1)}{4I} = \frac{mD(v_2 - v_1)}{4\frac{1}{12}MD^2} = \frac{3m}{M}\frac{(v_2 - v_1)}{D}$$

$$I = \frac{1}{12}MD^2$$



What if instead of a stick we have a thicker block so the bullet embeds itself in it?

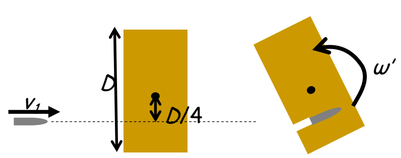


$$L_{z\,\text{before}} = mv_1\frac{D}{4} \qquad v = \frac{D}{4}\omega'_z$$

$$L_{z\,\text{after}} = mv\frac{D}{4} + I\omega'_z = m\left(\frac{D}{4}\right)^2\omega'_z + I\omega'_z = \left[m\left(\frac{D}{4}\right)^2 + I\right]\omega'_z$$

$$\omega'_z = \frac{mv_1\frac{D}{4}}{\frac{1}{16}mD^2 + \frac{1}{12}MD^2} = \frac{12m}{3m+4M}\frac{v_1}{D}$$

Total linear momentum $\vec{p}_{\text{total}}$ is **not** conserved, because $\vec{F}_{\text{net,ext}} \neq 0$



Example: Collapsing sphere

A spherical shell rotates about an axis through its center of mass. It has an initial radius R_i and angular speed ω_i. By applying a radial force, we can cause the sphere to collapse to $R_f = R_i/3$. What is the ratio of the final and the initial angular speed, ω_f/ω_i?

A. 1/9 B. 1/3 C. 1 D. 3 **E. 9**

The force to collapse the sphere is radial, so it produces zero torque.

Net torque = 0 → Angular momentum is conserved.

$L_i = I_i\omega_i$ $L_f = I_f\omega_f$

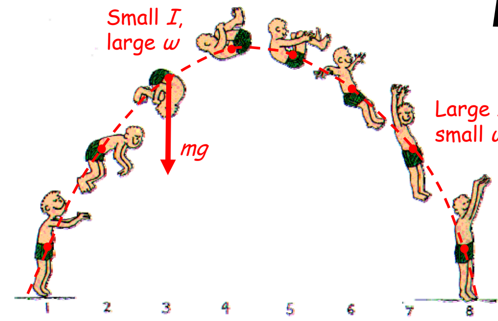
$$I_i\omega_i = I_f\omega_f \longrightarrow \frac{\omega_f}{\omega_i} = \frac{I_i}{I_f} = \frac{MR_i^2}{MR_f^2} = \frac{R_i^2}{R_f^2} = 9$$

DEMOs: Hoberman sphere & rotating stool

Example: Diver

Weight does not produce any torque about the CM of the diver

⟶ L is conserved



VIDEO: Divers

Torque and angular acceleration: the complete story

In all these situations (the Hoberman sphere, the rotating chair, the divers), the angular velocity changed. Where does the angular acceleration come from, if the net torque is zero?

$$\vec{\tau}_{ext} = \frac{d\vec{L}}{dt} = \frac{d(I\vec{\omega})}{dt} = I\frac{d\vec{\omega}}{dt} + \frac{dI}{dt}\vec{\omega}$$

$$\vec{\tau}_{ext} = I\vec{\alpha} + \frac{dI}{dt}\vec{\omega}$$

$\vec{\tau}_{ext} = I\vec{\alpha}$ only when I is constant

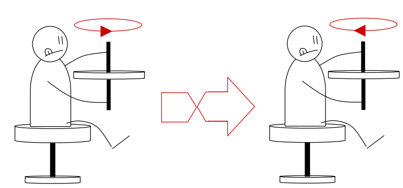
ACT: Conservation of L

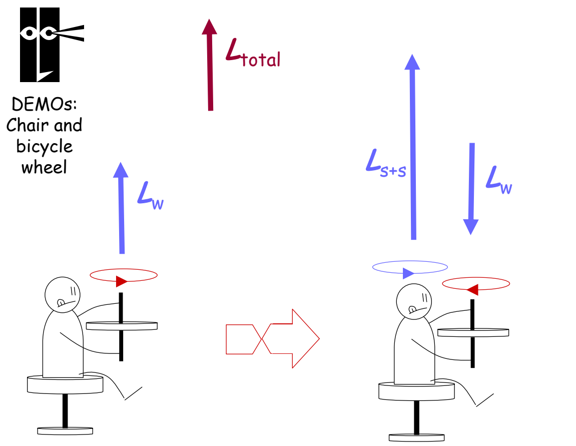
A student sits on a rotating stool and holds a rotating horizontal bicycle wheel by a rod through its axis. The stool is initially at rest. The student flips the axis of rotation of the wheel by 180°. What happens to the stool?

A. It rotates in the same direction as the wheel <u>after</u> the flip.

B. It rotates in the same direction as the wheel <u>before</u> the flip

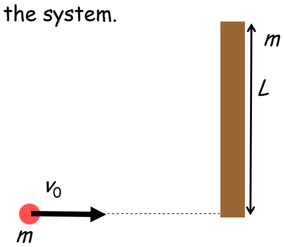
C. Nothing! Why would it rotate at all?





Advanced example: Collision with rotation

A ball of Playdough moving with speed v_0 hits a rod of equal mass on one of its ends. The rod is initially at rest and it begins moving on a horizontal, frictionless surface. The ball sticks to the rod after the collision. Describe the final motion of the system.



The motion will be something like this:

Run

A ball of Playdough moving with speed v_0 hits a rod of equal mass on one of its ends. The rod is initially at rest and it begins moving on a horizontal, frictionless surface. The ball sticks to the rod after the collision. Describe the final motion of the system.

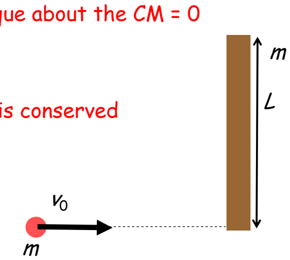
Net external force = 0 ⟶ P is conserved

The only non-zero external forces (weights, normals by surface) do not do torque about the CM of the system.

Net external torque about the CM = 0

↓

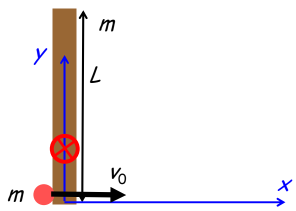
L about the CM is conserved



Center of mass of the system:

$x_{CM} = 0$ (neglect width of rod and size of ball)

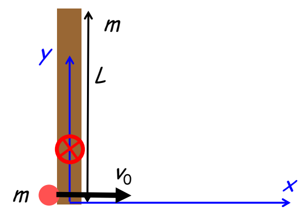
$$y_{CM} = \frac{m \times 0 + m\left(\frac{L}{2}\right)}{2m} = \frac{L}{4}$$



Conservation of $\boldsymbol{p}$:

$mv_0 = 2mv_{CM,x} \longrightarrow \boxed{v_{CM,x} = \frac{v_0}{2}}$

$0 = 2mv_{CM,y} \longrightarrow \boxed{v_{CM,y} = 0}$



Conservation of L_z about the CM of the system: $mv_0 \frac{L}{4} = I\omega$

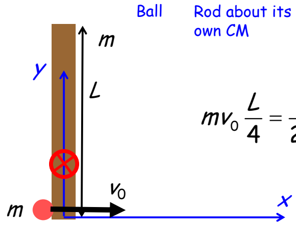
$$I = m\left(\frac{L}{4}\right)^2 + \frac{1}{12}mL^2 + m\left(\frac{L}{4}\right)^2 = \frac{5}{24}mL^2$$

Ball

Rod about its own CM

Parallel axis theorem correction for axis through CM of the system.

$$mv_0 \frac{L}{4} = \frac{5}{24}mL^2\omega \longrightarrow \boxed{\omega = \frac{6v_0}{5L}}$$



$\boxed{\vec{v}_{CM} = \frac{v_0}{2}\hat{i}}$ The CM moves in a straight line with constant speed…

$\boxed{\omega = \frac{6v_0}{5L}}$ … and the system rotates about the CM