Lecture 23

Work and power for rotations. Examples.

Work done by a torque (for pure rotations)

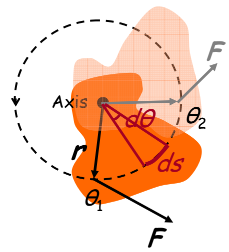
A force F acts on an object as it rotates from θ_1 to θ_2.

Work done along a small displacement:

$dW = \vec{F} \cdot d\vec{l} = F_{tan} ds = F_{tan} r\, d\theta = \tau_z d\theta$

Work from θ_1 to θ_2: $W = \int_{\theta_1}^{\theta_2} \tau_z d\theta$

If torque is constant, $W = \tau_z \, \Delta\theta$



Power done by a torque (for pure rotations)

Work done along a small displacement: $dW = \tau_z d\theta$

Instantaneous power: $P = \frac{dW}{dt} = \tau_z \frac{d\theta}{dt} = \tau_z \omega_z$

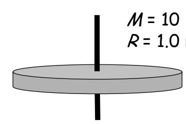
Average power: $\bar{P} = \frac{W}{\Delta t} = \tau_z \bar{\omega}_z$

Important! These expressions are only true for pure rotations (with a fixed axis of rotation, or in the frame of reference where the axis is at rest).

Otherwise, the path by the point where the force is applied is not a circle, as assumed in the previous slide!

Example: Work by friction

A uniform flywheel moving at 600 rpm comes to a stop after 1000 turns, mostly due to air resistance. What is the average torque produced by air?



$M = 10$ kg
$R = 1.0$ m

$\omega_0 = 600 \frac{\text{rev}}{\text{min}} \frac{1 \text{ min}}{60 \text{ s}} \frac{2\pi \text{ rad}}{1 \text{ rev}} = 20\pi$ rad/s

$I = \frac{1}{2} MR^2 = 5$ kg m^2

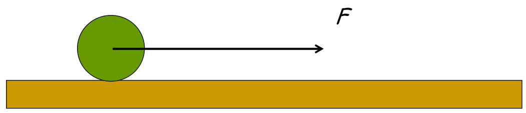
$\Delta\theta = 1000 \times 2\pi = 2000\pi$ rad

$$\tau_{z,average} = \frac{W}{\Delta\theta} = \frac{\Delta KE}{\Delta\theta} = \frac{0 - \frac{1}{2} I \omega_0^2}{\Delta\theta} = -1.6 \text{ Nm}$$

Example: Disk and string

A massless ideal string goes through the center a disk of mass M = 2 kg and radius R = 0.5 m and used to make it roll over a table by pulling on the string with a constant force F = 20 N. The cylinder rolls without slipping.

Find the speed of the cylinder after it has moved for d = 10 m.



A. 10.1 m/s B. 11.5 m/s C. 14.1 m/s D. 16.7 m/s E. 18.0 m/s

Only the pulling force is doing work: $W = Fd$

And static friction?

$$KE_f - KE_i = \frac{1}{2}Mv_{CM}^2 + \frac{1}{2}I_{disk}\omega^2 - 0 =$$

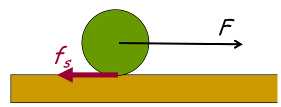
$$= \frac{1}{2}Mv_{CM}^2 + \frac{1}{2}\left(\frac{1}{2}MR^2\right)\left(\frac{v_{CM}}{R}\right)^2 = \frac{3}{4}Mv_{CM}^2$$

$$W = \Delta KE \longrightarrow Fd = \frac{3}{4}Mv_{CM}^2$$

$$v_{CM} = \sqrt{\frac{4Fd}{3M}} = 11.5 \text{ m/s}$$

Answer B

(If you forget K_{rot}, you get C)



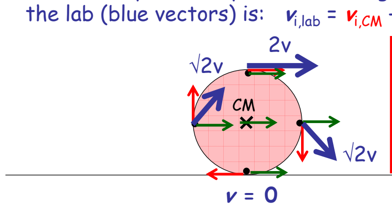
The point on which f_s acts is always different and always at rest: there is no displacement, so $W_{static\ friction} = 0$

In case you are not convinced…

The velocity relative to the CM of each point on the edge has magnitude $v = R\omega$ and direction as shown below (red vectors).

The velocity of the CM relative to the lab has magnitude v and direction as shown below (green vector).

The velocity of each point on the edge relative to the lab (blue vectors) is: $\mathbf{v}_{i,lab} = \mathbf{v}_{i,CM} + \mathbf{v}_{CM,lab}$



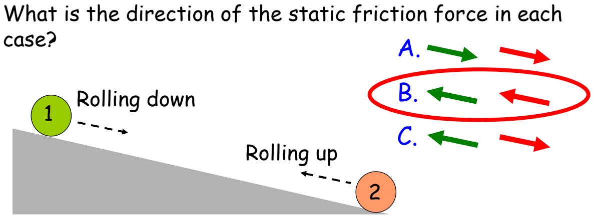
The contact point in a rolling object is instantaneously at rest

ACT: Figuring out friction

Cylinder 1 is released on an incline and rolls down (w/o slipping).

Cylinder 2 has an initial angular speed at the bottom of the ramp and starts rolling up (w/o slipping).

What is the direction of the static friction force in each case?

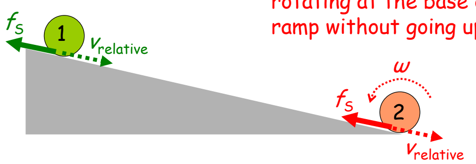


Friction must oppose relative motion.

How would the cylinders move in the absence of friction?

Cylinder 1 would slide down (no rotation)

Cylinder 2 would keep rotating at the base of the ramp without going up.

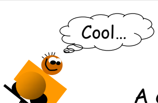


Or… look at the needed torque.

$\vec{\alpha}$ points into the page in both cases

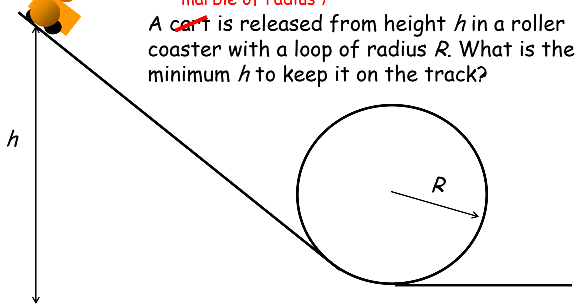
The weight and the normal produce zero torque about the CM.

To get the appropriate torque direction (into the page), f_S must point up in both cases.



Example: Loop-the-loop

A ~~cart~~ marble of radius r is released from height h in a roller coaster with a loop of radius R. What is the minimum h to keep it on the track?



Point B is the toughest point.

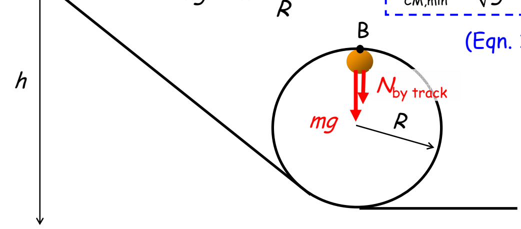
In order not to fall (ie, to keep the circular trajectory), the forces at B must provide the appropriate radial acceleration:

$$mg + N = m\frac{v_{CM}^2}{R}$$

The minimum velocity is fixed by $N = 0$:

$$mg = m\frac{v_{CM,min}^2}{R} \Rightarrow v_{CM,min} = \sqrt{gR}$$

(Eqn. 1)



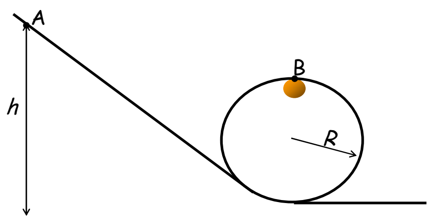
What is the speed at point B? $I_{\text{sphere about } CM} = \frac{2}{5}mr^2$

$E_A = E_B$

$$mgh + 0 = mg2R + \frac{1}{2}mv_{CM}^2 + \boxed{\frac{1}{2}I_{\text{sphere}}\omega^2}$$

New term

No slipping for $r \ll R$: $v_{CM} = r\omega$



$$mgh = mg2R + \frac{1}{2}mv_{CM}^2 + \frac{1}{2}\frac{2}{5}mr^2\frac{v_{CM}^2}{r^2}$$

$$gh = g2R + \frac{1}{2}\left(1 + \frac{2}{5}\right)v_{CM}^2$$

$$g(h - 2R) = \frac{7}{10}v_{CM}^2$$

$$v_{CM} = \sqrt{\frac{5}{7}2g(h - 2R)}$$

(Eqn. 2)

Let us put equations 1 and 2 together:

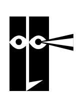
$$v_{CM,\min} = \sqrt{gR} \qquad v_{CM} = \sqrt{\frac{5}{7}2g(h - 2R)}$$

The minimum height is given by:

$$\sqrt{gR} = \sqrt{\frac{10}{7}g(h_{\min} - 2R)}$$

$$7R = 10h_{\min} - 20R$$

$$\boxed{h_{\min} = 2.7R}$$



DEMO: Loop the loop

(Without rotation the factor is 2.5)