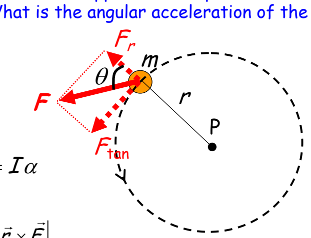
Lecture 22
Torque

An intuitive approach to torque

A particle of mass m is constrained by a massless rod of length r to move in circles about point P. A force F is applied on the particle. This is the only force applied on it. What is the angular acceleration of the system?



F_{tan} produces a tangential acceleration:

$F_{tan} = ma_{tan} \longrightarrow F_{tan} = m\alpha r$

$a_t = \alpha r$

$\longrightarrow F_{tan} r = \alpha m r^2 \longrightarrow \tau = I\alpha$

I

Torque $\tau = F_{tan} r = Fr\sin\theta = |\vec{r} \times \vec{F}|$

Directions: Both $\vec{\alpha}$ and $\vec{r} \times \vec{F}$ point out of the screen.

$\vec{\tau} = \vec{r} \times \vec{F}$ $\tau_z = I\alpha_z$

Newton's second law for rotations

For each force on a system, the torque depends on the point on which it is applied

$$\vec{\tau} = \vec{r} \times \vec{F}$$

The angular acceleration will be the result of adding all these torques:

$$\tau_{net,z} = \sum_{\text{all forces}} \tau_{i,z} = I\alpha_z$$

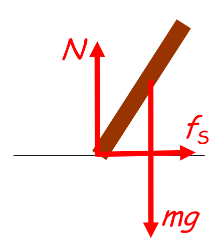
Note: In general, $\vec{\tau}_{net} \neq$ torque by the net force

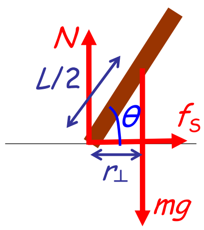
(What would be the $\vec{r}$?!?)

EXAMPLE: Falling tree

What is the angular acceleration of a tree as it falls down?

Model of a tree:
uniform rod of length L





Torque about the bottom of the tree (axis):

$$\tau_{net} = \tau_{mg} + \tau_N + \tau_f =$$

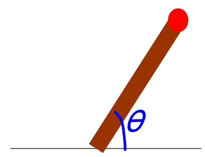
$$= mgr_\perp + 0 + 0$$

$$= mg\frac{L}{2}\cos\theta$$

Newton's 2nd law for rotations:

$$\tau_{net} = I\alpha, \quad \text{with } I = \frac{1}{3}mL^2$$

$$mg\frac{L}{2}\cos\theta = \frac{1}{3}mL^2\alpha \quad \Rightarrow \quad \boxed{\alpha = \frac{3g}{2L}\cos\theta}$$



A curious effect:

What is the tangential acceleration of the top of the tree?

$$a_t = L\alpha = \frac{3g}{2}\cos\theta$$

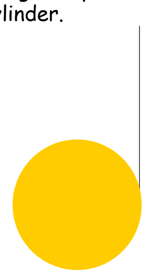
The top of the tree will have an acceleration larger than g when:

$$\frac{3g}{2}\cos\theta > g \quad \Rightarrow \quad \cos\theta > \frac{2}{3} \quad \Rightarrow \quad \boxed{\theta < 48.2^\circ}$$

DEMO:
Jumping ball

EXAMPLE: Yo-yo

A cylinder with a massless string wrapped around it is released as the free end of the string is kept fixed. Find the acceleration of the cylinder.



Newton's 2nd law for the translation of the CM:

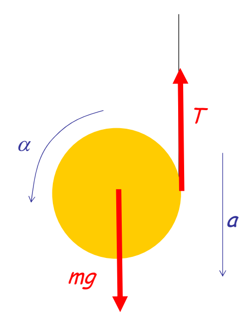
$$mg - T = ma$$

Newton's 2nd law for the rotation about the CM:

$$TR = I\alpha = \frac{1}{2}mR^2\alpha$$

String does not slip on the cylinder:

$$a = R\alpha$$



$$\begin{cases} mg - T = ma \\ TR = \frac{1}{2} mR^2 \alpha \\ a = R\alpha \end{cases} \qquad \begin{cases} mg - T = ma \\ T = \frac{1}{2} ma \end{cases}$$

$$mg - \frac{1}{2} ma = ma \longrightarrow \boxed{a = \frac{2}{3} g}$$

Note that $a < g$: The net force on the cylinder is less than mg.

Example: Real pulley

What is the angular acceleration of the pulley if R = 25.0 cm and M = 2.00 kg?

Direction: Into the page

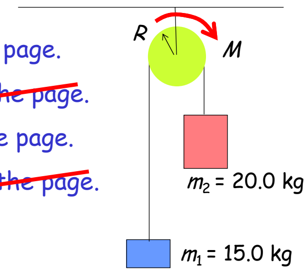
A. 196 rad/s^2 into the page.

B. 196 rad/s^2 ~~out of the page~~.

C. 5.45 rad/s^2 into the page.

D. 5.45 rad/s^2 ~~out of the page~~.

E. None of the above.



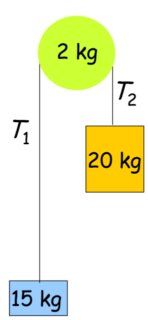
ACT: Real Atwood's machine

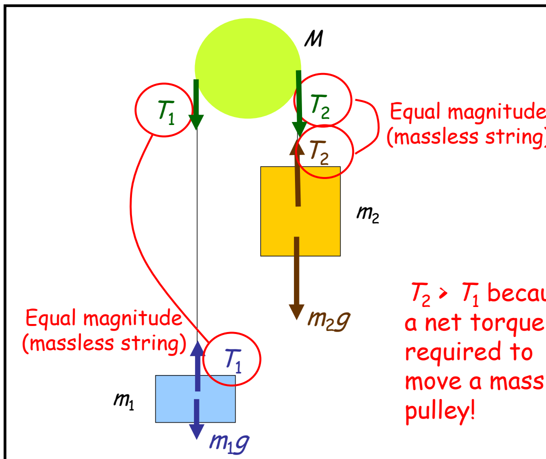
Compare the tensions on each side of the rope.

A. $T_1 < T_2$

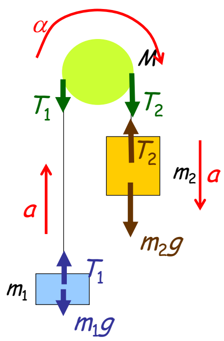
B. $T_1 = T_2$

C. $T_1 > T_2$





$T_2 > T_1$ because a net torque is required to move a massive pulley!



Newton's second law for every object:

Mass 1: $T_1 - m_1 g = m_1 a$

Mass 2: $m_2 g - T_2 = m_2 a$

Pulley: $T_2 R - T_1 R = I\alpha$

$I_{disk} = \frac{1}{2}MR^2$

+ String does not slip on the pulley: $a = R\alpha$

4 equations. 4 unknowns (a, α, T_1, T_2)

$$\begin{cases} T_1 - m_1 g = m_1 a \\ m_2 g - T_2 = m_2 a \\ T_2 R - T_1 R = I\alpha \\ a = R\alpha \end{cases} \longrightarrow \begin{cases} T_1 = m_1(a+g) \\ T_2 = m_2(g-a) \\ T_2 - T_1 = \frac{I}{R^2}a \end{cases}$$

$$m_2(g-a) - m_1(a+g) = \frac{I}{R^2}a$$

$$a = g\frac{m_2 - m_1}{m_1 + m_2 + \frac{I}{R^2}} \longrightarrow a = g\frac{m_2 - m_1}{m_1 + m_2 + \frac{M}{2}}$$

$I_{disk} = \frac{1}{2}MR^2$

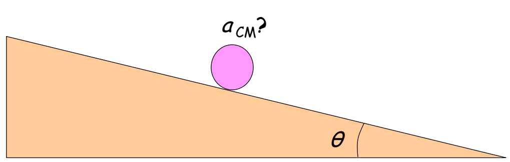
$$\alpha = \frac{a}{R} = \frac{g}{R}\frac{m_2 - m_1}{m_1 + m_2 + \frac{M}{2}} = \frac{9.81}{0.25} \times \frac{20-15}{20+15+1} = 5.45 \text{ rad/s}^2$$

Into the page

EXAMPLE: Cylinder rolling down an incline

A cylinder of mass M and radius R rolls down an incline of angle θ with the horizontal.

If the cylinder rolls without slipping, what is its acceleration?



Newton's 2nd law for translation of the CM:

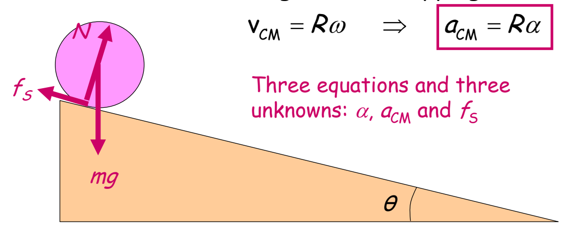
$Mg\sin\theta - f_S = Ma_{CM}$

Newton's 2nd law for rotation about the CM:

$f_S R = I_{CM}\alpha$ (with $I_{CM} = \frac{1}{2}MR^2$)

Rolling without slipping

$v_{CM} = R\omega \Rightarrow a_{CM} = R\alpha$



Three equations and three unknowns: α, a_{CM} and f_S

$$\begin{cases} Mg\sin\theta - f_S = Ma_{CM} \\ f_S R = I_{CM}\alpha \\ a_{CM} = R\alpha \end{cases} \longrightarrow \begin{cases} Mg\sin\theta - Ma_{CM} = f_S \\ f_S = I_{CM}\dfrac{a_{CM}}{R^2} \end{cases}$$

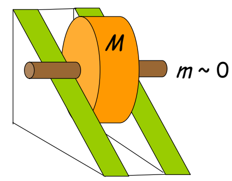
$$Mg\sin\theta - Ma_{CM} = I_{CM}\frac{a_{CM}}{R^2}$$

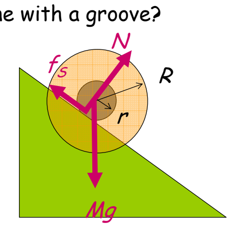
$$a_{CM} = \frac{Mg\sin\theta}{M + \dfrac{I_{CM}}{R^2}} = \frac{Mg\sin\theta}{M + \dfrac{MR^2}{2R^2}} = \frac{2}{3}g\sin\theta$$

$$I_{\text{solid cylinder}} = \frac{1}{2}MR^2$$

(compare to $g\sin\theta$, the results for a sliding object)

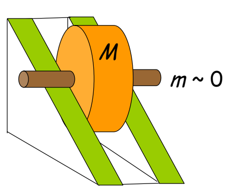
And what if instead we use a disk of radius R and mass M that mounted on a massless shaft of radius $r \ll R$ and have the whole thing roll down an incline with a groove?





$$\begin{cases} Mg\sin\theta - f_S = Ma_{CM} \\ f_S r = I_{CM}\alpha \\ a_{CM} = r\alpha \end{cases} \longrightarrow a_{CM} = \frac{Mg\sin\theta}{M + \dfrac{I_{CM}}{r^2}}$$

$$a_{CM} = \frac{Mg\sin\theta}{M + \dfrac{I_{CM}}{r^2}}$$



$$I_{CM} = \frac{1}{2}MR^2$$

DEMO:
Disk on shaft on incline

$$a_{CM} = \frac{Mg\sin\theta}{M + \dfrac{MR^2}{2r^2}} = \boxed{\frac{1}{1 + \dfrac{R^2}{2r^2}}}\, g\sin\theta$$

Very small if $R \gg r$!

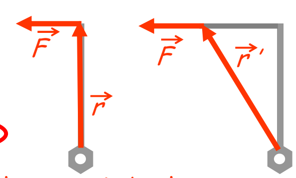
ACT: Loosening a stuck nut

You have to loosen a stuck nut. In your toolbox, there are two wrenches. One is a "normal" wrench (figure 1) and the other has a perpendicular extension at the end (figure 2). If you apply a horizontal force in both cases as shown, which of these two arrangements will be more efficient?

A. The normal one

B. The one with the extension.

C. It doesn't matter.



The torque is the same in both cases.

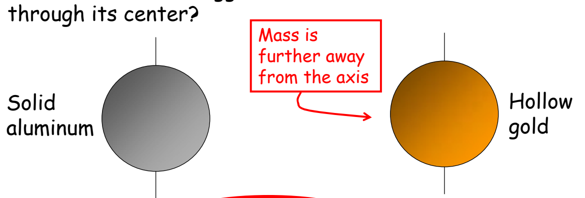
$|\vec{r} \times \vec{F}| = |\vec{r}' \times \vec{F}| = F\,d$ (out of the page)

(cross product picks up the perpendicular parts)

ACT: Spheres

Two spheres have the same radius and equal masses. One is made of solid aluminum and the other is a hollow shell of gold.

Which one has the biggest moment of inertia about an axis through its center?



A. Solid Al

B. Hollow Au

C. Both the same