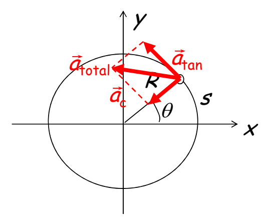
Lecture 21

Rotations of a Rigid Body. Moment of inertia

Review of circular motion

Description in terms of angular quantities (in radians!): θ; $\omega = \dfrac{d\theta}{dt}$; $\alpha = \dfrac{d\omega}{dt}$

Relation to linear quantities: $s = R\theta$ $\quad v = R\omega$ $\quad a_{tan} = R\alpha$



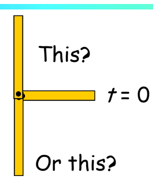
Centripetal acceleration

$$a_c = R\omega^2 = \frac{v^2}{R}$$

Angular velocity

Example: A rod on the plane of the page rotates about an axis through one of its ends at 0.25 rpm. If the initial position is as shown, what is its position at t = 1 min?

We need to indicate whether the motion is CW or CCW!

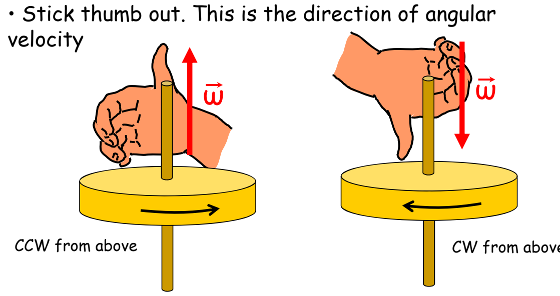


Angular velocity $\vec{\omega}$:

- Magnitude $\omega = \dfrac{d\theta}{dt} = \dfrac{v}{R}$
- Direction: perpendicular to the plane of motion and in the direction given by the RHR.

The right-hand rule (again)

- Curl fingers of right hand in the direction of motion.
- Stick thumb out. This is the direction of angular velocity



ACT: Wall clock

Let $\vec{\omega}_L$ and $\vec{\omega}_S$ be the angular velocities of the large and the small handles of the wall clock in this room. What is the direction of the difference $\vec{\omega}_S$ - $\vec{\omega}_L$?

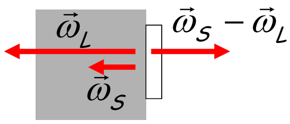
$\omega_L > \omega_S$

Both $\vec{\omega}_L$ and $\vec{\omega}_S$ point into the wall.

A. Into the wall.

B. Out of the wall.

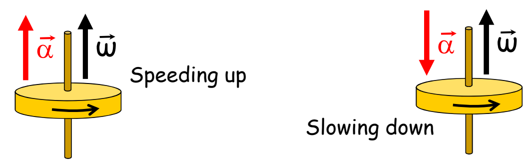
C. Nothing, it's zero.



Angular acceleration

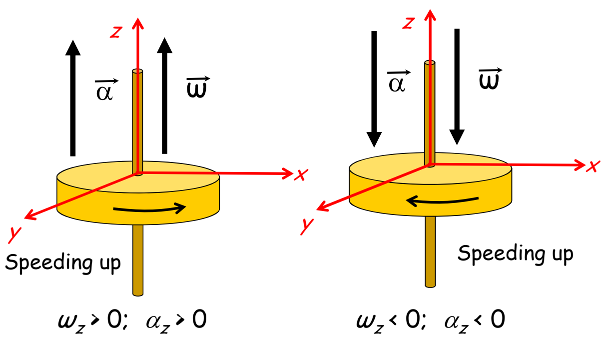
Angular acceleration $\vec{\alpha}$:

- Magnitude $\alpha = \dfrac{d\omega}{dt} = \dfrac{a_{tan}}{R}$
- Direction: perpendicular to the plane of motion, parallel (antiparallel) to $\vec{\omega}$ if the system is speeding up (slowing down).



When the plane of rotation is constant, we can always choose that plane to be the *xy* plane.

Then, all that matters is the sign of ω_z and α_z.



ACT: Angular acceleration

A ball rolls across the floor and then starts up a ramp as shown below. In what direction does the angular acceleration vector point when the ball is on the ramp?

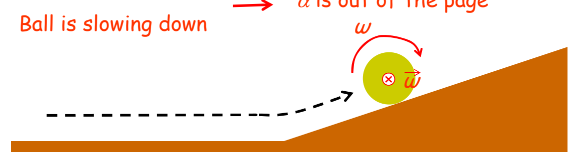
A. Down the ramp

B. Into the page

C. Out of the page

$\vec{\omega}$ is into the page

Ball is slowing down → $\vec{\alpha}$ is out of the page



Rigid body

A rigid body is a system where internal forces hold each part in the same relative position.

Motion which a rigid body:

* Motion of the center of mass -in response to an external force.
* Rotations about the CM.

Or

* Rotations about a fixed axis

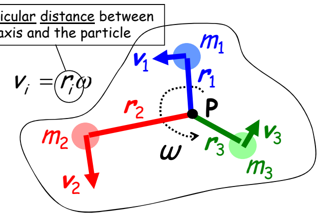
Energy of Rotational Motion

Consider a rigid body being rotated around an axis perpendicular to the page and through point P. The body can be broken down into n particles.

$$K_{tot} = \sum_{i=1}^{n} \tfrac{1}{2} m_i v_i^2 = \sum_{i=1}^{n} \tfrac{1}{2} m_i (r_i\omega)^2 = \tfrac{1}{2}\left(\sum_{i=1}^{n} m_i r_i^2\right)\omega^2$$

Perpendicular distance between the axis and the particle

$$v_i = r_i\omega$$



Moment of inertia (about this axis): $I = \sum_{i=1}^{n} m_i r_i^2$

$$K_{rotational} = \tfrac{1}{2} I\omega^2$$

Moment of Inertia

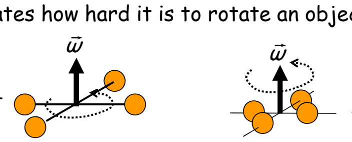
DEMO: Bars

$$I = \left(\sum_{i=1}^{n} m_i r_i^2\right)$$

Units: kg m²

The moment of inertia is to rotation what mass is to translation.

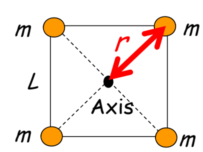
→ Indicates how hard it is to rotate an object.



It depends on the position and orientation of the axis.

Example: Moment of inertia of a square of side L made with four identical particles of mass m and four massless rods.

a. For rotations about an axis perpendicular to the square, through the center of the square.



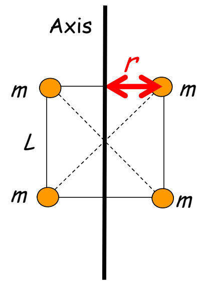
Perpendicular distance between each mass and the axis:

$$r = \frac{\sqrt{2}}{2}L$$

$$I = 4\left[m\left(\frac{\sqrt{2}}{2}L\right)^2\right] = 2mL^2$$

Example: Moment of inertia of a square of side L made with four identical particles of mass m and four massless rods.

b. For rotations about an axis in the plane of the square, through the center of the square.



Perpendicular distance between each mass and the axis:

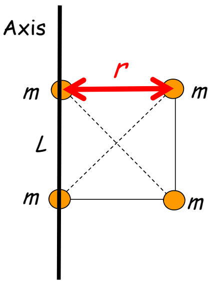
$$r = \frac{L}{2}$$

$$I = 4\left[m\left(\frac{L}{2}\right)^2\right]$$

$$= mL^2$$

Example: Moment of inertia of a square of side L made with four identical particles of mass m and four massless rods.

b. For rotations about an axis in the plane of the square, through one side of the square.



Perpendicular distance between each mass and the axis:

$r = L$ for two of the particles;

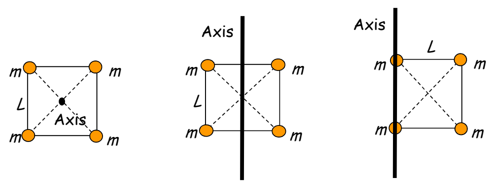
$r = 0$ for the other two.

$$I = 2\left[mL^2\right] + 0$$

$$= 2mL^2$$

Recap:

Example: Moment of inertia of a square of side L made with four identical particles of mass m and four massless rods.



$I = 2mL^2$ $I = mL^2$ $I = 2mL^2$

The moment of inertia depends on the position and orientation of the axis.

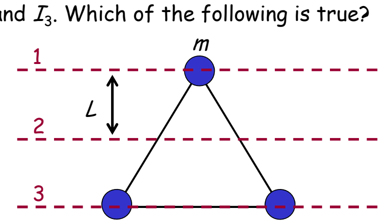
ACT: Triangle

Three identical balls are connected with three identical, rigid, massless rods. The moments of inertia about axes 1, 2 and 3 are I_1, I_2 and I_3. Which of the following is true?

A. $I_1 > I_2 > I_3$

B. $I_1 > I_3 > I_2$

C. $I_2 > I_1 > I_3$



$I_1 = m(2L)^2 + m(2L)^2 = 8mL^2$

$I_2 = mL^2 + mL^2 + mL^2 = 3mL^2$

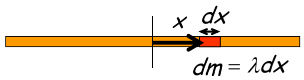
$I_3 = m(2L)^2 = 4mL^2$

Continuous Mass Distribution

Break the object into infinitesimal masses *dm* and do an integral:

$$I = \int r^2 dm$$

EXAMPLE: Uniform rod of length *L* and mass *M* for rotations about the perpendicular axis through its center.



Density: $\lambda = \frac{M}{L}$

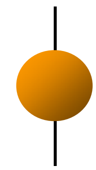
$$I = \int x^2 dm = \int_{-L/2}^{L/2} x^2 \lambda dx = \lambda \frac{1}{3}\left[\left(\frac{L}{2}\right)^3 - \left(-\frac{L}{2}\right)^3\right] = \lambda \frac{L^3}{12} = \frac{M}{L}\frac{L^3}{12} = \frac{1}{12}ML^2$$

Note: It's the same for a rectangle! Width did not play any role.

Some other important moments of inertia

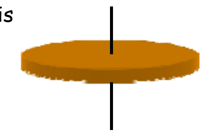
Solid sphere of mass M and radius R, about the axis through its center:

$$I = \frac{2}{5}MR^2$$



Solid disk (or cylinder) of mass M and radius R, about the perpendicular axis through its center:

$$I = \frac{1}{2}MR^2$$



More: See book or formula sheet.

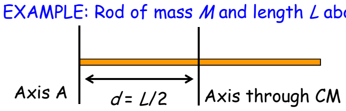
Parallel-axis theorem

The moment of inertia of a system for an axis parallel to another axis that goes through the center of mass is:

$$I_{parallel} = I_{CM} + Md^2$$

d = distance between both axes.

EXAMPLE: Rod of mass *M* and length *L* about the axis through one end:



$$I_{axis\ CM} = \frac{1}{12}ML^2$$

$$I_{axis\ A} = I_{axis\ CM} + Md^2 = \frac{1}{12}ML^2 + M\left(\frac{L}{2}\right)^2 = \frac{1}{3}ML^2$$

Back to the kinetic energy:

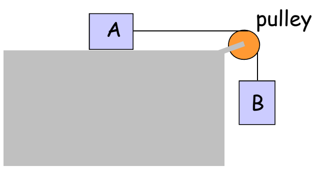
$$K_{rotational} = \tfrac{1}{2} I\omega^2$$

Everything we already know about energy is still true!

- Conservation of energy
- Work-kinetic energy theorem
- …

Example: Real pulley

Two 3.0-kg boxes is attached to a light rope that passes over a pulley (uniform disk of 2.0 kg, radius 20 cm) as shown below. There is no friction between the pulley and its axis or between the box and the table. The system is released from rest. Find the speed of box B when it has fallen 1.5 m.



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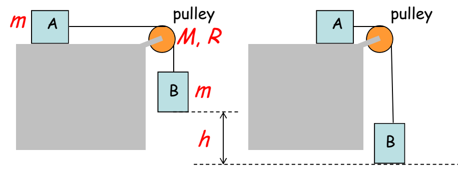


m = 3.0 kg
M = 2.0 kg
R = 20 cm
h = 1.5 m

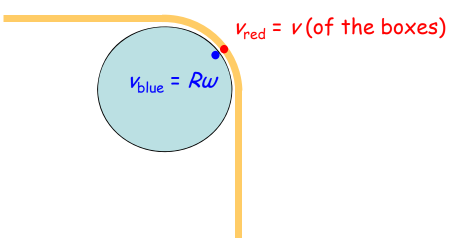
The net work is due to the weight of box B, so E is conserved.

$$mgh + 0 = 0 + KE_{trans,A} + KE_{trans,B} + KE_{rot,pulley}$$

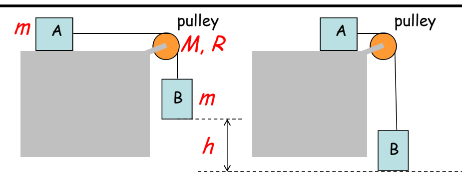
$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{1}{2}MR^2\right)\omega^2$$

Two unknowns (v, ω)!

If the string does not slip on the pulley, the blue point (on the edge of the pulley) and the red point (on the string) move together:



No slipping: $v = R\omega$



m = 3.0 kg
M = 2.0 kg
R = 20 cm
h = 1.5 m

$$mgh + 0 = 0 + KE_{trans,A} + KE_{trans,B} + KE_{rot,pulley}$$

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{1}{2}MR^2\right)\omega^2$$

no slipping: $R\omega = v$

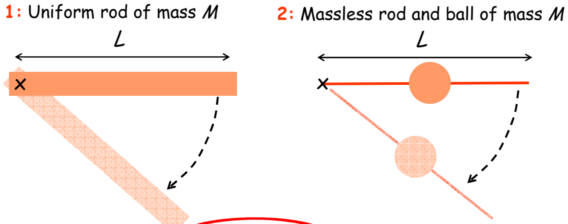
$$mgh = \frac{v^2}{2}\left(2m + \frac{M}{2}\right)$$

$$v = \sqrt{\frac{4m}{4m+M}gh} = \sqrt{\frac{4(3.0\text{ kg})}{4(3.0\text{ kg}) + 2.0\text{ kg}}(9.8\text{ m/s}^2)(1.5\text{ m})} = 3.5\text{ m/s}$$

ACT: Pivoting rods

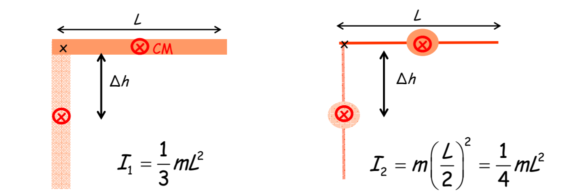
Both systems can pivot about one end, and they are released from the horizontal position. Use conservation of energy to determine which system is moving faster when it reaches the vertical position.

1: Uniform rod of mass M **2:** Massless rod and ball of mass M



A. System 1 B. System 2 C. Same for both

1: Uniform rod of mass M **2:** Massless rod and ball of mass M



$$I_1 = \frac{1}{3}mL^2 \qquad I_2 = m\left(\frac{L}{2}\right)^2 = \frac{1}{4}mL^2$$

Same gravitational potential difference (same Δh) → Same final $KE = \frac{1}{2}I\omega^2$ + $I_1 > I_2$ → $\omega_1 < \omega_2$

Motion of a system of particles

Motion of a system of particles can be broken down to:

1. Motion of each particles relative to the CM
2. Motion of the CM relative to the lab

In particular, for the kinetic energy:

$$K_{\text{system,lab}} = \sum_i \frac{1}{2} m_i v_{i,\text{lab}}^2 = \sum_i \frac{1}{2} m_i (\vec{v}_{i,CM} + \vec{v}_{CM,\text{lab}})^2$$

$$= \frac{1}{2}\sum_i m_i v_{i,CM}^2 + \frac{1}{2}2\left(\sum_i m_i \vec{v}_{i,CM}\right)\vec{v}_{CM,\text{lab}} + \frac{1}{2}\left(\sum_i m_i\right) v_{CM,\text{lab}}^2$$

$$= \frac{1}{2}\sum_i m_i v_{i,CM}^2 + \frac{1}{2}M v_{CM,\text{lab}}^2 = K_{\text{system},CM} + K_{CM,\text{lab}}$$

$\sum m_i \vec{v}_{i,CM} = M\vec{v}_{CM,CM} = 0$

Velocity of the CM relative to the CM

$$K_{\text{system,lab}} = K_{\text{system},CM} + K_{CM,\text{lab}}$$

Rotation & Translation

For any complex system: $K_{\text{parts,lab}} = K_{\text{parts},CM} + K_{CM,\text{lab}}$

For a rigid body, the motion of the parts relative to the CM can only be rotation:

$$K_{\text{parts},CM} = \frac{1}{2}\sum_i m_i v_{i,CM}^2 = \frac{1}{2}\left(\sum_i m_i r_{i,CM}^2\right)\omega^2 = \frac{1}{2}I_{CM}\omega^2$$

Thus the motion of a rigid body is:

rotation about the center of mass: + translation of the center of mass

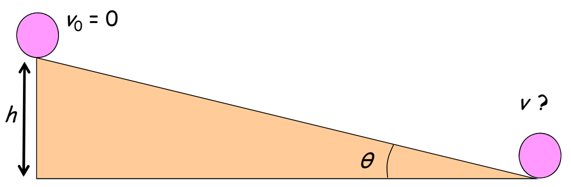
$$K_{\text{total}} = K_{\text{rotational}} + K_{\text{translational}}$$

$$K_{rot} = \frac{1}{2}I_{CM}\omega^2 \qquad K_{trans} = \frac{1}{2}Mv_{CM}^2$$

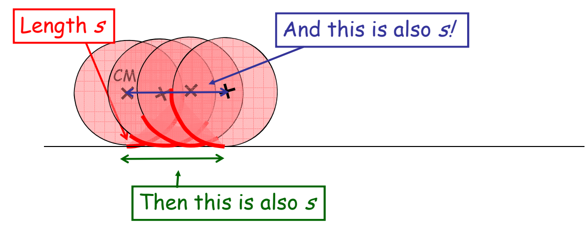
EXAMPLE: Cylinder rolling down an incline

A cylinder of mass *M* and radius *R* is placed at rest on top of an incline of height *h* and angle *θ* with the horizontal.

If the cylinder rolls without slipping, what is its speed when it reaches the bottom of the incline?



First: What is "rolling without slipping"?



Rotated arch: $s = R\Delta\theta$

Displacement of the CM: $\Delta x_{CM} = s = R\,\Delta\theta$

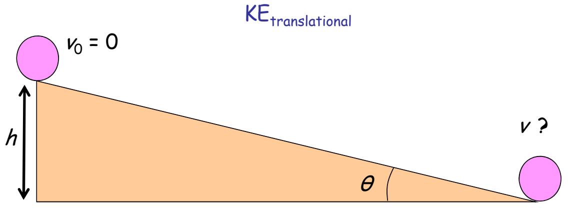
$$\Delta x_{CM} = R\Delta\theta \xrightarrow{\frac{d}{dt}} \boxed{v_{CM} = R\omega} \xrightarrow{\frac{d}{dt}} \boxed{a_{CM} = R\alpha}$$

Only conservative forces (weight) are doing work, so mechanical energy is conserved.

$$E_{top} = Mgh + 0$$

$$E_{bottom} = 0 + \boxed{\frac{1}{2}Mv^2} + \boxed{\frac{1}{2}I\omega^2}$$

KE$_{translational}$ (blue box), KE$_{rotational}$ (red box)



Only conservative forces (weight) are doing work, so mechanical energy is conserved.

$$E_{top} = Mgh + 0$$

$$E_{bottom} = 0 + \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$$

$$Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$$

Rolling without slipping: $v = R\omega$

$$\longrightarrow \quad Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}I\left(\frac{v}{R}\right)^2$$

$$Mgh = \frac{1}{2}\left(M + \frac{I}{R^2}\right)v^2$$

$$v = \sqrt{\frac{2gh}{1 + \frac{I}{MR^2}}}$$

$$I_{cylinder} = \frac{1}{2}MR^2$$

$$v = \sqrt{\frac{2gh}{1 + \frac{1}{2}}} = \sqrt{\frac{4}{3}gh} = 1.155\sqrt{gh}$$

What if it is not a cylinder, but a solid sphere of mass M and radius *R*?

$$I_{\text{sphere}} = \frac{2}{5}MR^2$$

$$v = \sqrt{\frac{2gh}{1+\frac{I}{MR^2}}} = \sqrt{\frac{2gh}{1+\frac{2}{5}}} = \sqrt{\frac{10}{7}gh} = 1.195\sqrt{gh}$$

Or a hollow cylinder of mass M and radius *R*?

$$I_{\text{sphere}} = MR^2$$

$$v = \sqrt{\frac{2gh}{1+\frac{I}{MR^2}}} = \sqrt{\frac{2gh}{1+1}} = \sqrt{gh}$$

DEMO:
Three different
objects down a ramp