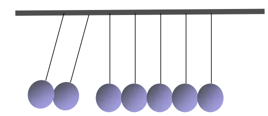
Lecture 20

Center of Mass

ACT: Newton's craddle

Consider a row of adjacent steel-ball pendula. If <u>two</u> balls on the left are pulled to a certain height h and released, what happens?



A. One ball rises on the right, but higher than h.

B. Two balls rise on the right to height h.

C. A ball rises on each side to height h (i.e., one of the initial balls bounces back)

A. One ball rises on the right, but higher than h.

We could have $2mv_i = mv_f$, so $v_f = 2v_i$.
But then kinetic energy would not be conserved:

$KE_i = 2\frac{1}{2}mv_i^2 = mv_i^2$ $\qquad$ $KE_f = \frac{1}{2}mv_f^2 = 2mv_i^2$

B. Two balls rise on the right at height h.

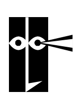
$2mv_i = 2mv_f$, so $v_f = v_i$

$KE_i = 2\,\frac{1}{2}mv_i^2 = mv_i^2$ $\qquad$ $KE_f = 2\,\frac{1}{2}mv_f^2 = mv_i^2$ $\qquad$ **Ok!**

C. A ball will rise on each side to height h (i.e., one of the initial balls bounces back)

Same height means $|v_i| = |v_f|$. But this violates conservation of momentum:

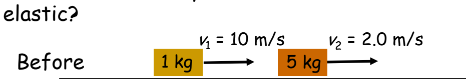
$p_i = 2mv_i$ $\qquad$ $p_f = mv_i - mv_i = 0$



DEMO: Newton's craddle

Example: Elastic collision

What is the velocity of each block if the collision is elastic?



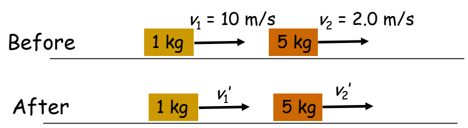
A. 2.0 m/s, 10 m/s

B. 5.3 m/s, 2.7 m/s

C. -5.3 m/s, 2.7 m/s

D. -3.3 m/s, 2.7 m/s

E. -3.3 m/s, 4.7 m/s



$$\begin{aligned} m_1v_1 + m_2v_2 &= m_1v_1' + m_2v_2' \\ v_1 - v_2 &= -(v_1' - v_2') \end{aligned} \longrightarrow \begin{aligned} 10 + 10 &= v_1' + 5v_2' \\ 10 - 2 &= -(v_1' - v_2') \end{aligned} \longrightarrow \begin{aligned} v_1' &= 20 - 5v_2' \\ v_1' &= v_2' - 8 \end{aligned}$$

$$20 - 5v_2' = v_2' - 8 \longrightarrow v_2' = \frac{28}{6} = 4.7 \text{ m/s}$$

$$v_1' = v_2' - 8 \longrightarrow v_1' = -3.3 \text{ m/s}$$

Answer: E

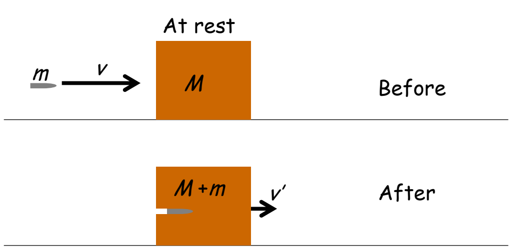
Example: Ballistic pendulum

Let's go back to the bullet and block example...

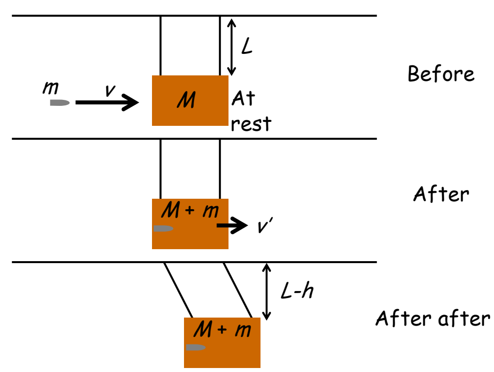
$p_{\text{total, initial}} = mv + 0$

$p_{\text{total, final}} = (M + m)v'$ $\longrightarrow$ $mv = (M + m)v'$

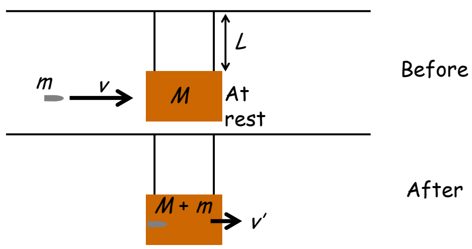
$$v' = \frac{m}{M + m}v$$



Now we'll do the same thing with the block hanging from two strings of length L.



Between "before" and "after", the collision is exactly the same as for the block on the table.



Linear momentum is conserved. $\longrightarrow$ $v' = \dfrac{m}{M+m} v$

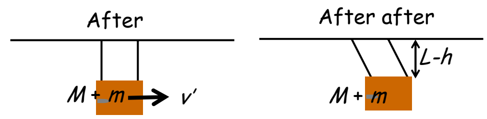
Between "after" and "after-after", momentum is not conserved ($F_{ext} \neq 0$). But only conservative forces are doing work, so E is conserved.

$$\frac{1}{2}(M+m)v'^2 + 0 = 0 + (M+m)\,gh \longrightarrow h = \frac{v'^2}{2g}$$

$$\longrightarrow \boxed{h = \frac{v^2}{2g}\left(\frac{m}{M+m}\right)^2}$$

Note that $(M+m)gh = \dfrac{v^2}{2}\left(\dfrac{m^2}{M+m}\right) = \dfrac{1}{2}mv^2\left(\dfrac{1}{1+\dfrac{M}{m}}\right) < \dfrac{1}{2}mv^2$

(some energy is lost in the collision)



ACT: Ballistic pendulum

The projectile is a ball with two sides. One is smooth and the other has two spikes, so the ball sticks to the wooden block. In which case will the block move higher?

A. When the smooth side impacts the block.

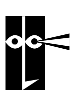
Larger momentum tranfer

B. When the spikes side impacts the block.

C. It's the same in both cases.

It's the same as the bowling pin ACT!

DEMO: Ballistic pendulum



Center of mass

The real-world is not made of point-like objects…

What is the "position" of an extended object?

For a system of n particles at positions $\vec{r}_1$, $\vec{r}_2$, etc, the position of the center of mass of the system is defined as the average of all the position weighted with the masses of the particles:

$$\vec{r}_{cm} = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i} = \frac{\sum_i m_i \vec{r}_i}{M}$$

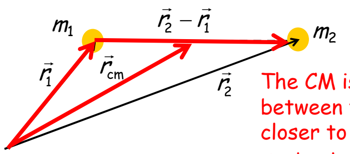
Another example of a weighted average:

If a course grade is 4/10 midterm grade and 6/10 final grade, and you get the following scores:

Midterm: 50/100
Final: 100/100

$$\text{course grade} = \frac{4}{10}50\% + \frac{6}{10}100\% = 80\%$$

EXAMPLE: Two masses:



The CM is always in the line between the two particles, and closer to the more massive one.

And… the position of the CM is independent of the reference frame.

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$

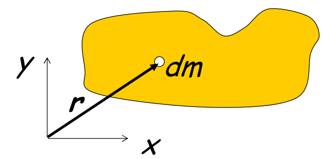
$$= \frac{(m_1 + m_2 - m_2)\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$

$$= \vec{r}_1 + \frac{m_2}{m_1 + m_2}(\vec{r}_2 - \vec{r}_1) \qquad 0 \le \frac{m_2}{m_1 + m_2} \le 1$$

Continuous mass distributions

Sum over particles ⟶ Integrals

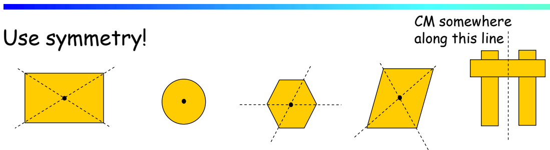
Break the object down to infinitessimal dm:



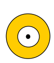
$$\vec{r}_{cm} = \frac{\int r\,dm}{\int dm} = \frac{\int r\,dm}{M}$$

Some basic properties of the CM

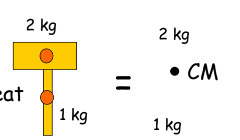
Use symmetry!



The CM need not be inside the object!



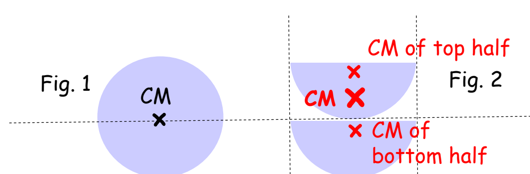
For a system composed of many shapes or parts, first condense each part to its CM and then treat each CM as a pointlike particle



ACT: Two disks

The disk shown in figure 1 is uniform and has its CM at the center. Suppose the disk is cut in half and its pieces arranged as shown in figure 2.

Where is the CM of (2) compared to the CM of (1)?



A. Higher. B. Lower. C. At the same level.

Velocity of the CM

$$\frac{d}{dt}\left(\vec{r}_{cm} = \frac{\sum_i m_i \vec{r}_i}{M}\right) \longrightarrow \boxed{\vec{v}_{cm} = \frac{\sum_i m_i \vec{v}_i}{M}}$$

Rearrange the expression:

$$M\vec{v}_{cm} = m_1\vec{v}_1 + m_2\vec{v}_2 + \cdots + m_n\vec{v}_n$$
$$= \vec{p}_1 + \vec{p}_2 + \cdots + \vec{p}_n$$
$$= \vec{P}_{total}$$

$$\boxed{\vec{P}_{total} = M\vec{v}_{cm}}$$

Acceleration of the CM

$$\frac{d}{dt}\left(\vec{v}_{cm} = \frac{\sum_i m_i \vec{v}_i}{M}\right) \longrightarrow \boxed{\vec{a}_{cm} = \frac{\sum_i m_i \vec{a}_i}{M}}$$

Rearrange the expression:

$$M\vec{a}_{cm} = m_1\vec{a}_1 + m_2\vec{a}_2 + \cdots + m_n\vec{a}_n$$
$$= \vec{F}_{net,1} + \vec{F}_{net,2} + \cdots + \vec{F}_{net,n}$$
$$= \vec{F}_{net\ external}$$

Internal forces come in 3rd law pairs and cancel out

$$\boxed{\vec{F}_{net\ ext} = M\vec{a}_{cm}}$$

Also:

$$\vec{F}_{net\ ext} = M\vec{a}_{cm} = M\frac{d\vec{v}_{cm}}{dt} = \frac{d\vec{P}_{total}}{dt}$$

We already knew that…

Motion of the CM

$$\boxed{\vec{P}_{total} = M\vec{v}_{cm}} \qquad \boxed{\vec{F}_{net\ ext} = M\vec{a}_{cm}}$$

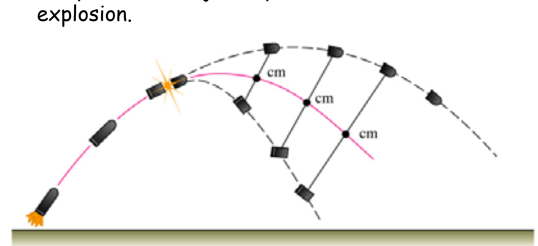
The CM is a good "representation" of the extended object.

Internal forces among the parts may change the velocities and accelerations of the parts, but the velocity of the CM of a system remains constant unless it is acted on by an external force.

Conservation of linear momentum = Constant velocity of center of mass

Trajectory of the CM

When a shell explodes, the CM keeps moving along the parabolic trajectory the shell had before the explosion.

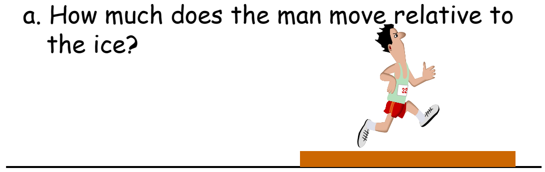


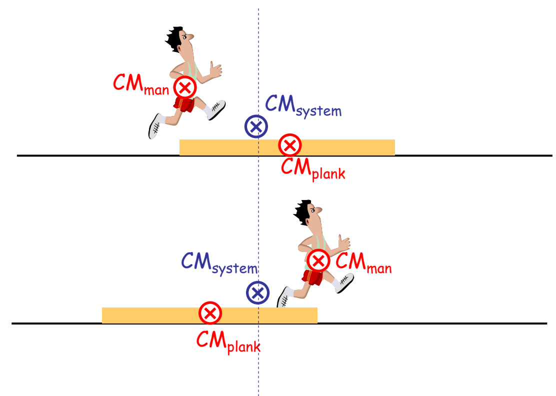
DEMO: Shape

EXAMPLE: Running on a plank

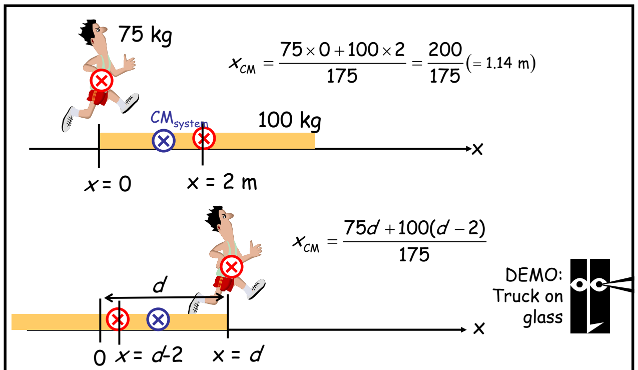
A 75-kg man is standing on a 100-kg plank that is 4 m long. The plank is at rest on a frozen lake. The man starts running on the plank at 5 km/h (relative to the plank).

a. How much does the man move relative to the ice?





The center of mass of the system does not move.



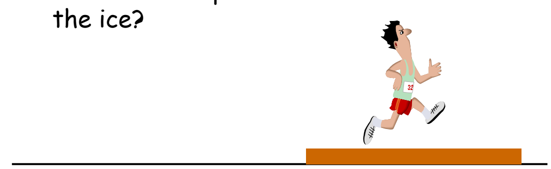
$$x_{CM} = \frac{75 \times 0 + 100 \times 2}{175} = \frac{200}{175} (= 1.14 \text{ m})$$

$$x_{CM} = \frac{75d + 100(d-2)}{175}$$

$$75d + 100(d-2) = 200 \qquad d = 2.3 \text{ m}$$

A 75-kg man is standing on a 100-kg plank that is 4 m long. The plank is at rest on a frozen lake. The man starts running on the plank at 5 km/h (relative to the plank).

b. What is the speed of the man relative to the ice?



$$v_{CM,ice} = 0 \quad \Rightarrow \quad m v_{man,ice} - M v_{plank,ice} = 0 \quad \Rightarrow \quad v_{plank,ice} = \frac{m}{M} v_{man,ice}$$

$$v_{man,ice} = v_{man,plank} - v_{plank,ice}$$

$$v_{man,ice} = v_{man,plank} - \frac{m}{M} v_{man,ice}$$

$$v_{man,ice} = \frac{v_{man,plank}}{1 + \frac{m}{M}} = \frac{5\ \text{km/h}}{1 + \frac{75\ \text{kg}}{100\ \text{kg}}} = 2.9\ \text{km/h}$$