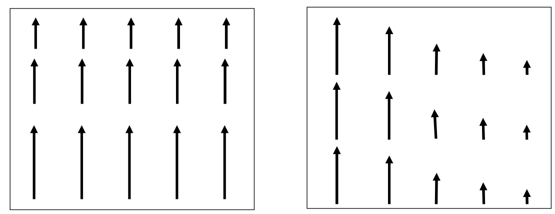
Lecture 17

Energy Diagrams

ACT: Force field

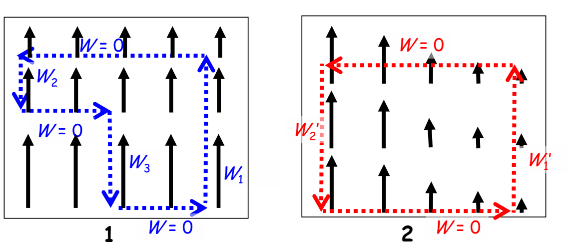
The pictures below show force vectors at different points in space for two different forces. Which of these forces is conservative?



1 2

A. 1 B. 2 C. Both

Work along a closed trajectories (loops).



$|W_2 + W_3| = |W_1|$

$\rightarrow \quad W_{loop} = 0$

$|W_2'| > |W_1'|$

$\rightarrow \quad W_{loop} \neq 0$

Relation between U and F (1D)

For a conservative force (in 1D),

$$W = \int_{\text{initial}}^{\text{final}} F_x \cdot dx = -\Delta U \longrightarrow \boxed{U = -\int F_x \cdot dx + \text{constant}}$$

$$\updownarrow$$

$$\boxed{F_x = -\frac{dU}{dx}}$$

Examples:

$U = mgy \quad \Rightarrow \quad F_y = -\frac{dU}{dy} = -mg$

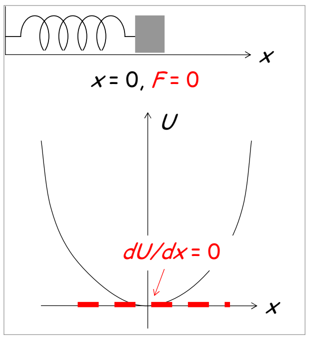
$U = \frac{1}{2}kx^2 \quad \Rightarrow \quad F_x = -\frac{dU}{dx} = -kx$

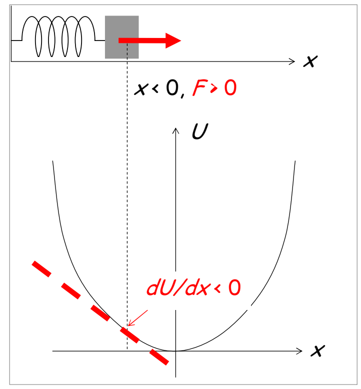
The force is minus the slope of the $U(x)$ curve.

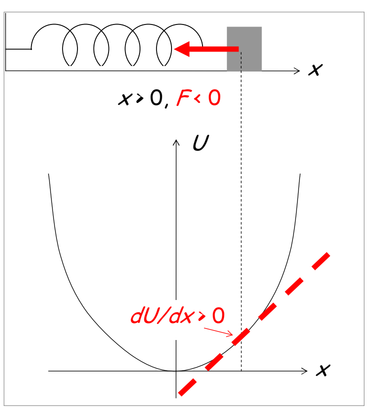
Example: Spring

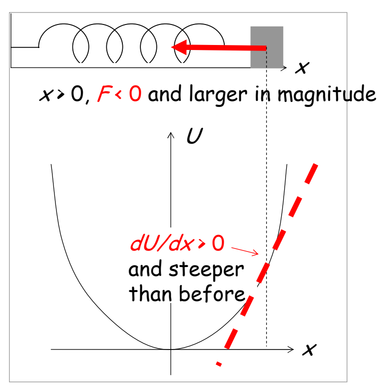
Potential energy of a box attached to a spring on a horizontal, frictionless table.

$$U(x) = \frac{1}{2}kx^2$$





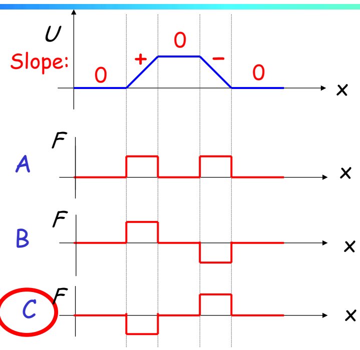




The force always points “downhill”!!!

ACT: U vs F

Which of the force versus position graphs matches the potential energy function shown in blue?



Force = – slope

Relation between U and F (3D)

1D: $F_x = -\dfrac{\partial U}{\partial x}$ ("downhill")

3D: $\vec{F} = -\vec{\nabla} U$ (minus the gradient of U)

("downhill in the steepest direction")

$F_x = -\dfrac{\partial U}{\partial x},\ F_y = -\dfrac{\partial U}{\partial y},\ F_z = -\dfrac{\partial U}{\partial z}$ in cartesian coordinates

$F_r = -\dfrac{\partial U}{\partial r}$ for the radial component in spherical coordinates

Example

Find the force exerted at point P (0,1,2) m if the potential energy associated with the force is:

$$U(\vec{r}) = \left(3xy - 4x^2 - yz^3\right)\ \text{J}$$

$F_x = -\dfrac{\partial U}{\partial x} = -(3y - 8x)$ $\qquad F_{x,P} = -(3-0) = -3$

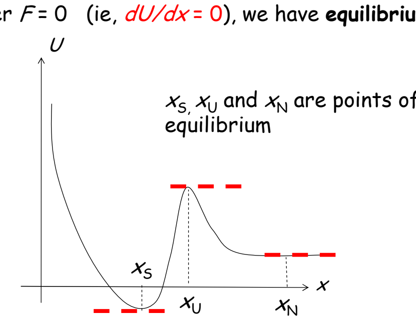
$F_y = -\dfrac{\partial U}{\partial y} = -\left(3x - z^3\right)$ $\qquad F_{y,P} = -(0-8) = 8$

$F_z = -\dfrac{\partial U}{\partial z} = -\left(-3yz^2\right)$ $\qquad F_{z,P} = -(-3 \times 1 \times 4) = 12$

$$\vec{F}_P = \left(-3\hat{i} + 8\hat{j} + 12\hat{k}\right)\ \text{N}$$

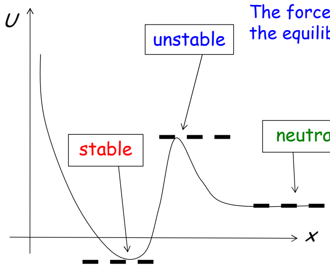
Equilibrium

Whenever $F = 0$ (ie, $dU/dx = 0$), we have **equilibrium**.



Stable/unstable/neutral equilibrium

What happens if the particle moves some small dx away from the equilibrium point?



The force pulls it away from the equilibrium point.

The force remains zero, so the particle stays at the new position, which is also an equilibrium position.

The force brings it back to the equilibrium point.

DEMO: Equilibrium

Energy Diagrams

It is a plot of U versus x, where the mechanical energy can also be included.

Energy diagrams provide a very useful visual representation of a conservative force.

Key features of the force are easily revealed.

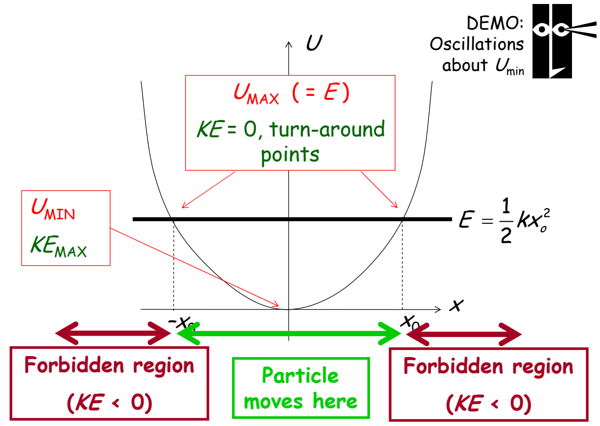
- Force points "downhill"
- Maximum = unstable equilibrium point
- Minimum = stable equilibrium point
- Turning points : Wherever $E = U$ (so $K = 0$)

Mechanical energy is related to the initial conditions.

Example 1: A box attached to a spring on a horizontal, frictionless table is released at $x = x_0$ from rest.

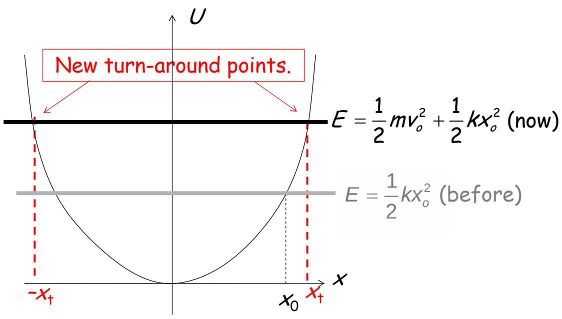
$$E = KE + U = 0 + \frac{1}{2}kx_o^2$$

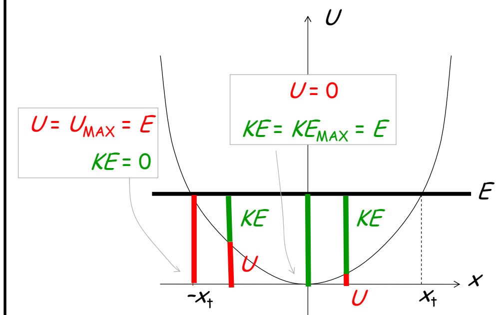
DEMO: Oscillations about U_{min}



Example 2: The box is brought to $x = x_0$ and pushed, so its initial velocity is v_0.

$$E = KE + U = \frac{1}{2}mv_o^2 + \frac{1}{2}kx_o^2$$

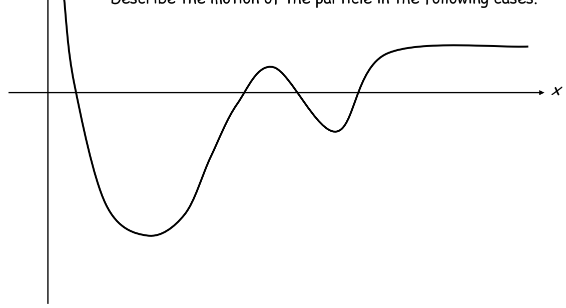




How much kinetic/potential energy does the system have at every point?

Example: Potential with two pits.

A particle is subjected to the force associated with this potential. No other forces are exerted on the particle. Describe the motion of the particle in the following cases.



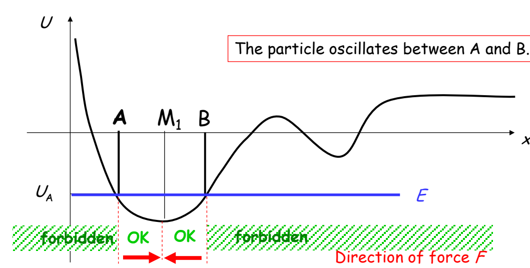
1. The particle is released from rest at point A.

$E = K + U$ From the initial conditions, $E = 0 + U_A$

At M_1, U is minimum, so K (and speed) is maximum

At x_B, $U = E$, so K (and speed) is zero → turn around point

The particle is forbidden from $x < x_A$ or $x > x_B$ ($K < 0$)

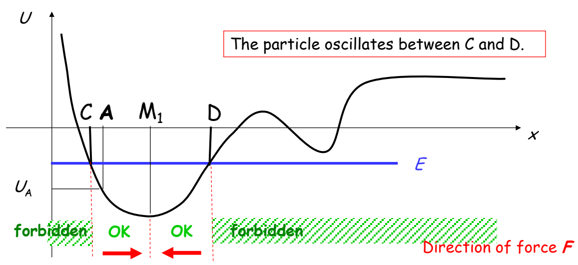


2. The particle is released at point *A* with a small* initial velocity v_0i.

From the initial conditions, $E = \frac{1}{2}mv_0^2 + U_A > U_A$ (*but not too much larger)

At M_1, *U* is minimum, so *K* (and speed) is maximum

The turn-around points are defined by *K* = 0, so *U* = *E*: points C and D.

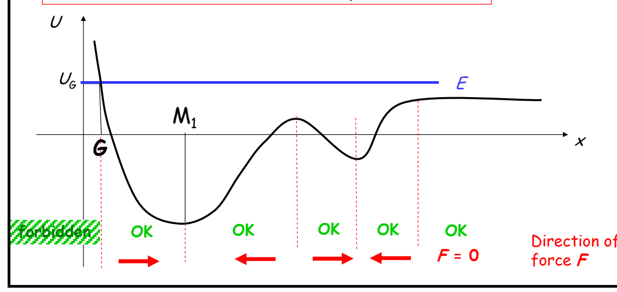


3. The particle is released from rest at point *G*.

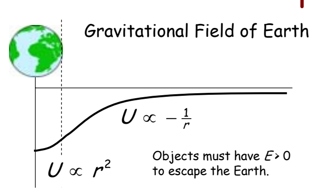
From the initial conditions, $E = U_G$

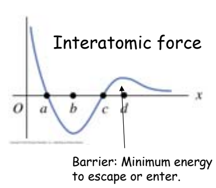
At M_1, *U* is minimum, so *K* (and speed) is maximum

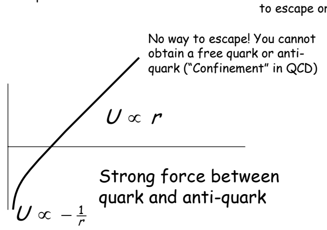
The particle keeps moving in the +x direction (no oscillations) and never comes back ("escapes").



"Escape"







ACT!

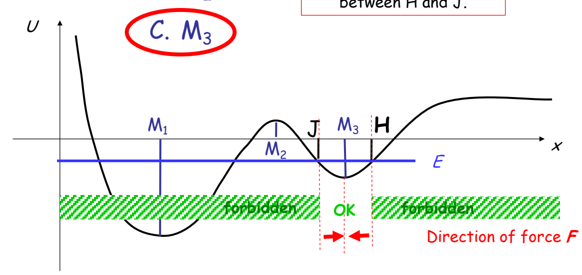
4. The particle is released from rest at point H. The particle has maximum speed at point:

A. M_1

B. M_2

C. M_3

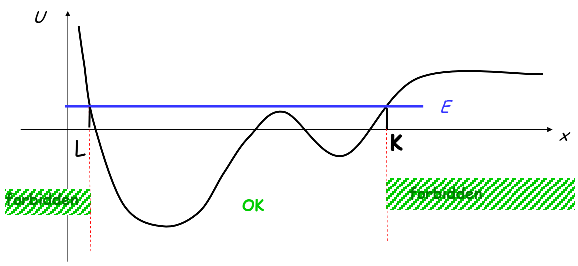
The particle oscillates between H and J.



5. The particle is released from rest at point K.

From initial conditions, $E = U_K$

The particle oscillates between K and L.



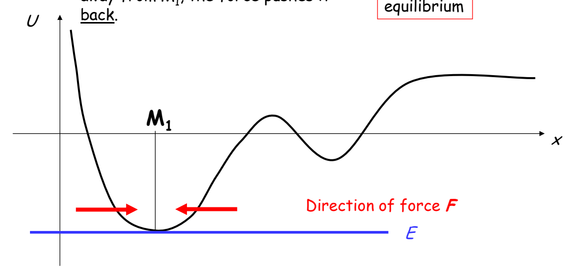
6. The particle is released from rest at point M_1.

From initial conditions, $E = U_{M1}$

Force = 0 with $v = 0$ → Equilibrium

If someone pushes the particle slightly away from M_1, the force pushes it back.

Stable equilibrium



7. The particle is released from rest at point M_2.

From initial conditions, $E = U_{M2}$

Force = 0 with $v = 0$ → Equilibrium

If someone pushes the particle slightly away from M_1, the force pushes it further away.

Unstable equilibrium