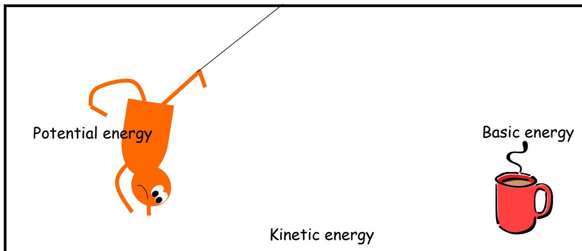
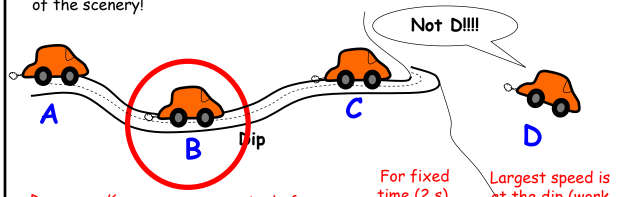


Lecture 15

Energy Conservation; Potential Energy

ACT: Car Accident?

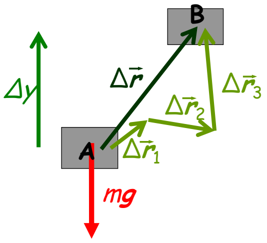
You are speeding down the road without a care in the world and just as you are approaching a sharp curve in the road above a scenic overlook you notice that your brakes are soft due to leaking brake fluid. From past experience you estimate that you have about 2 s of braking left. Before the dangerous curve there is a dip in the road. Which is the best place to use your limited brake? You must slow down before the curve or you will become part of the scenery!



Decrease K as much as possible → Need large $|W|$ → Apply force during long distance → For fixed time (2 s), longest distance when large speed → Largest speed is at the dip (work by gravity increases K)

Work done by gravity

A block of mass m is lifted from the floor (A) to a table (B) through two different trajectories. Find the work done by gravity.



$$\boxed{W} = m\vec{g}\cdot\Delta\vec{r} = -mg\Delta y$$

$$\boxed{W} = m\vec{g}\cdot\Delta\vec{r}_1 + m\vec{g}\cdot\Delta\vec{r}_2 + m\vec{g}\cdot\Delta\vec{r}_3$$
$$= m\vec{g}\cdot\left(\Delta\vec{r}_1+\Delta\vec{r}_2+\Delta\vec{r}_3\right)$$
$$= m\vec{g}\cdot\Delta\vec{r}$$
$$= -mg\Delta y$$

Work by gravity does not depend on the path

Gravitational potential energy

The work done by gravity does not depend on the path, it only depends on the vertical displacement Δy, or on the initial and final y:

$$W = -mg\Delta y$$

We can ALWAYS write this work as (minus) the change in some function $U(\mathbf{r})$ that depends on position (not on path):

$$W = -\left(U_f - U_i\right) = -\Delta U$$

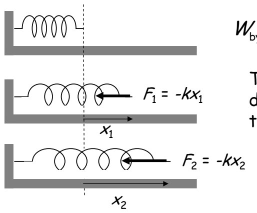
U= potential energy

Gravitational potential energy: $U = mgy + \text{constant}$

There is always room for an arbitrary constant, because what matters is ΔU

Elastic Potential Energy (Spring)

What is the work done by a spring as the tip is pulled from x_1 to x_2?



$$W_{\text{by spring}} = -\int_{x_1}^{x_2} kx\,dx = -\left(\frac{1}{2}kx_2^2 - \frac{1}{2}kx_1^2\right)$$

This can also be written as (minus) the difference of a potential function at the initial and final points x_1 and x_2:

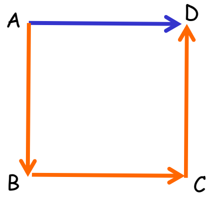
$$W_{\text{by spring}} = -\left(U(x_2) - U(x_1)\right)$$

Elastic potential energy: $U = \frac{1}{2}kx^2 + \text{constant}$

Can work always be written in terms of a potential energy change?

NO!

Example: A box is dragged along a rough horizontal surface through two paths AD and ABCD:



$W_{\text{friction,AD}} = -df_k$

$W_{\text{friction,ABCD}} = -3df_k$

Does not depend on initial and final points only.

The work done by friction CANNOT be written as a potential difference.

Conservative and non-conservative forces

The work done by a conservative force does not depend on the trajectory.

- A potential energy function can be defined.

Examples: Gravity, spring

Non-conservative force = force that is not conservative.

- The work done by a non-conservative force depends on the trajectory.
- A potential energy function cannot be defined.

Examples: Kinetic friction

Conservation of Mechanical Energy

In a system where only conservative forces are doing work, we can rewrite the WKE theorem:

$$W_{\text{net}} = \Delta K \xrightarrow{W_{\text{net}} = -\Delta U} -\Delta U = \Delta K \longrightarrow \Delta(K+U) = 0$$

Definition of Mechanical Energy: $E = K + U$

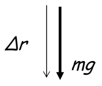
Under the conditions above, mechanical energy is conserved: $\Delta E = 0$ or $E_{\text{initial}} = E_{\text{final}}$

Example: Free fall

A ball is dropped from a height *h*. If the initial speed is 0 and we ignore air resistance, what is the speed of the ball as it hits the ground?

We can use kinematics or… the WKE theorem… or conservation of energy.

WKE



Work done by gravity: *mgh*

$$W = \Delta K$$

$$mgh = \frac{1}{2}mv^2$$

$$v = \sqrt{2gh}$$

Conservation of energy

The only force doing work is gravity, so mechanical energy is conserved.

$$E_{\text{initial}} = E_{\text{final}}$$

$$K_{\text{initial}} + U_{\text{initial}} = K_{\text{final}} + U_{\text{final}}$$

$$0 + mgh = \frac{1}{2}mv^2 + 0$$

Choice: *U* = 0 at ground level

$$mgh = \frac{1}{2}mv^2$$

$$v = \sqrt{2gh}$$

Two ways of looking at the same problem.

An object of mass *m* is lifted by a person from the floor to height *h* and dropped.

1. Lifting: $W_{\text{net}} = W_{\text{gravity}} + W_{\text{person}} = 0$ ($v_{\text{initial}} = v_{\text{final}} = 0$)

   Drop: $W_{\text{net}} = W_{\text{gravity}} = mgh > 0$ (thus *K* increases)

2. Lifting: $W_{\text{person}} > 0$. Person is adding energy which is stored as gravitational potential energy $mgh > 0$.

   Drop: The potential energy is converted into kinetic energy (thus *K* increases)

ACT: Up an incline

A box of mass *m* and initial speed v_0 = 10 m/s moves up a frictionless incline angled 30°. How high does the box go before it begins sliding down?

- A. 2 m
- [✓ B. 5 m]
- C. 10 m

$$E = K + U$$

$$E_A = \frac{1}{2}mv_0^2 + 0$$

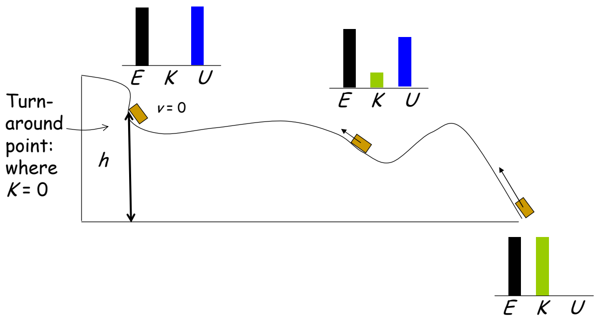
$$E_B = 0 + mgh$$

$$E_A = E_B \Rightarrow \frac{1}{2}mv_0^2 = mgh$$

$$\Rightarrow h = \frac{v_0^2}{2g} = \frac{(10\text{ m/s})^2}{2(10\text{ m/s}^2)} = 5\text{ m}$$

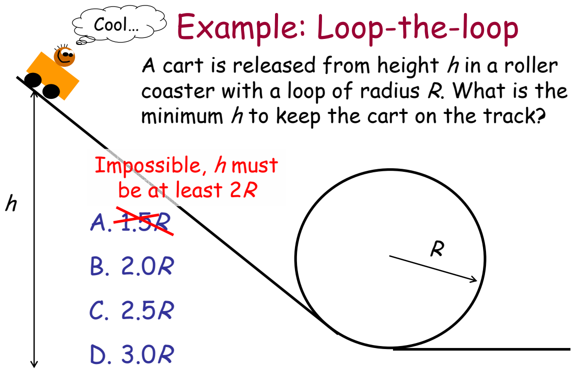
Only gravity does work (the normal is perpendicular to the motion), so mechanical energy is conserved.

The really nice thing is, we can apply the same thing to any "incline":



Example: Loop-the-loop

A cart is released from height *h* in a roller coaster with a loop of radius *R*. What is the minimum *h* to keep the cart on the track?



Impossible, *h* must be at least 2*R*

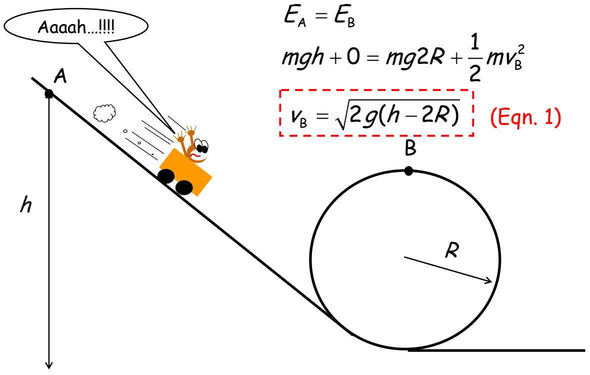
- A. ~~1.5*R*~~
- B. 2.0*R*
- C. 2.5*R*
- D. 3.0*R*
- E. 4.0*R*

Point B is the toughest point. What is the speed there?

$$E_A = E_B$$

$$mgh + 0 = mg2R + \frac{1}{2}mv_B^2$$

$$\boxed{v_B = \sqrt{2g(h-2R)}} \quad \text{(Eqn. 1)}$$

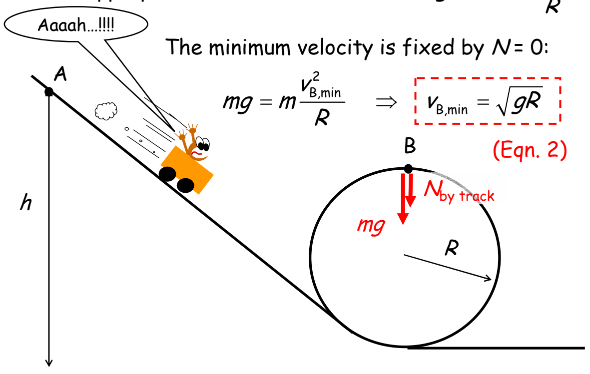


In order not to fall (ie, to keep the circular trajectory), the forces at B must provide the appropriate radial acceleration: $mg + N = m\frac{v_B^2}{R}$

The minimum velocity is fixed by *N* = 0:

$$mg = m\frac{v_{B,\min}^2}{R} \Rightarrow \boxed{v_{B,\min} = \sqrt{gR}}$$

(Eqn. 2)



Let us put equations 1 and 2 together:

$$v_B = \sqrt{2g(h-2R)} \qquad v_{B,\min} = \sqrt{gR}$$

The minimum height is given by:

$$\sqrt{gR} = \sqrt{2g(h_{\min} - 2R)}$$

$$R = 2h_{\min} - 4R$$

$$\boxed{h_{\min} = \frac{5}{2}R}$$

Answer C

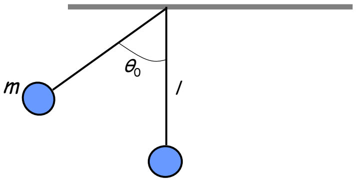
DEMO:
Loop the loop

EXAMPLE: Pendulum

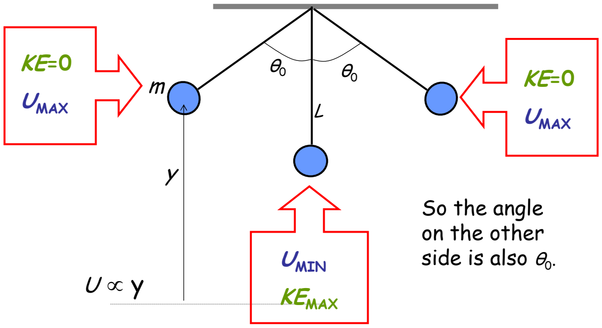
Consider a pendulum of length l and mass m, that is released from rest at an angle θ_0.

a. What is the maximum angle that the pendulum will reach on the other side?

b. What is the maximum speed of the pendulum?



Only weight is doing work, so it's a situation where mechanical energy $E = KE + U$ is conserved.



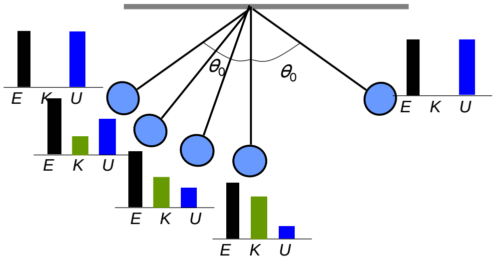
KE=0
U_{MAX}

KE=0
U_{MAX}

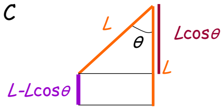
U_{MIN}
KE_{MAX}

So the angle on the other side is also θ_0.

Potential energy U is transformed into kinetic energy K. And viceversa.



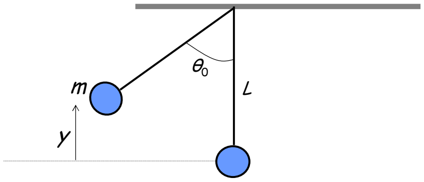
$U = mgy + C\ \ = mgL(1\text{-}\cos\theta) + C$



Let's take $U(\theta = 0) = 0$

$\Rightarrow\ C = 0$

$U = mgL(1\text{-}\cos\theta)$

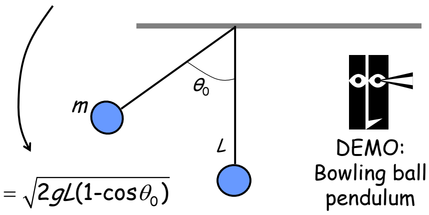


$E = mgL(1-\cos\theta) + mv^2/2 = \text{constant}$

Initially: $E = mgL(1-\cos\theta_0) + 0$

At the bottom: $E = 0 + mv^2/2$

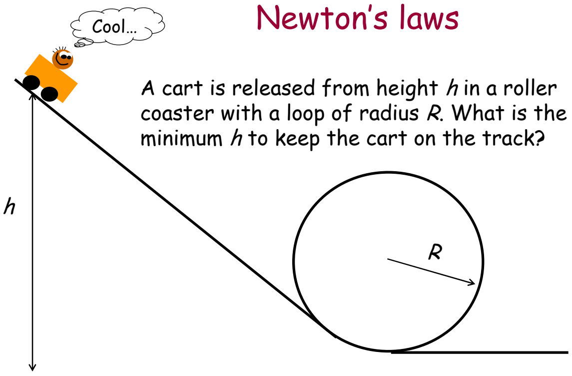
$0 + mv^2/2 = mgL(1-\cos\theta_0) + 0$



$v = \sqrt{2gL(1-\cos\theta_0)}$

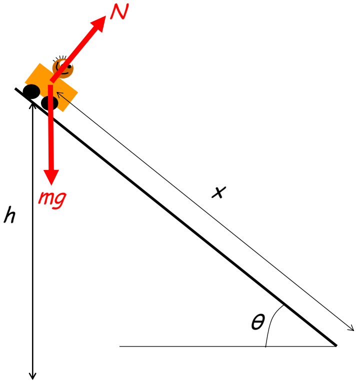
DEMO:
Bowling ball pendulum

Appendix: Loop-the-loop with Newton's laws



A cart is released from height *h* in a roller coaster with a loop of radius *R*. What is the minimum *h* to keep the cart on the track?

Part 1: The straight section



$$mg\sin\theta = ma$$
$$a = g\sin\theta$$

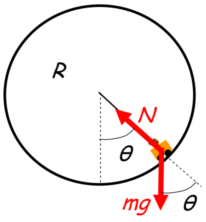
Speed at the bottom:

$$v_{\text{bottom}}^2 = 2ax$$
$$= 2a\frac{h}{\sin\theta}$$
$$v_{\text{bottom}} = \sqrt{2\frac{h}{\sin\theta}g\sin\theta}$$
$$= \sqrt{2gh}$$

Part 2: The circular section

In the radial direction, at any given *θ*: $N - mg\cos\theta = m\frac{v^2}{R}$

In the tangential direction, at any given *θ*: $mg\sin\theta = ma_t$



This is circular motion with a non-uniform angular acceleration!

In terms of angular quantities:

$$\begin{cases} N - mg\cos\theta = mR\omega^2 \\ g\sin\theta = R\alpha \end{cases}$$

$$\begin{cases} N - mg\cos\theta = mR\left(\dfrac{d\theta}{dt}\right)^2 & (1) \\ g\sin\theta = R\dfrac{d^2\theta}{dt^2} & (2) \end{cases}$$

Differential equations for θ
(at least not coupled)

On equation (2), use this trick: $\frac{d^2\theta}{dt^2} = \frac{d\omega}{dt} = \frac{d\omega}{d\theta}\frac{d\theta}{dt} = \frac{d\omega}{d\theta}\omega$

$$\omega\frac{d\omega}{d\theta} = \frac{g}{R}\sin\theta$$

(now it's a first order differential equation)

$$\omega\ d\omega = \frac{g}{R}\sin\theta d\theta$$

integrate

$$\frac{1}{2}\left(\omega^2 - \omega_0^2\right) = \frac{g}{R}\left(\cos\theta - \cos 0^\circ\right)$$

where ω_0 is the angular speed at the bottom, see part 1:

$$\omega_0 = \frac{v_{\text{bottom}}}{R} = \frac{\sqrt{2gh}}{R}$$

$$\omega^2 = \frac{2gh}{R^2} + \frac{2g}{R}\left(\cos\theta - 1\right) = \frac{2g}{R}\left(\frac{h}{R} + \cos\theta - 1\right)$$

At the top, θ = 180°:

$$\omega_{\text{top}}^2 = \frac{2g}{R}\left(\frac{h}{R} - 2\right) \quad (3)$$

From equation (1), and for θ = 180°:

$$N + mg = mR\omega_{\text{top}}^2$$

$\Big\downarrow$ Minimum speed $\Leftrightarrow$ N = 0

$$mg = mR\omega_{\text{top, min}}^2 \quad (4)$$

(3) and (4): $\frac{2g}{R}\left(\frac{h_{\text{min}}}{R} - 2\right) = \frac{g}{R} \longrightarrow h_{\text{min}} = \frac{5}{2}R$