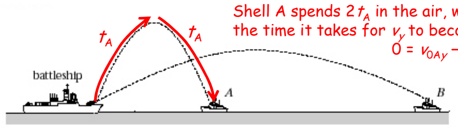
Lecture 6

Circular Motion

ACT: Battleship

A battleship simultaneously fires two shells at enemy ships. If the shells follow the parabolic trajectories shown, which ship gets hit first?



Shell A spends $2t_A$ in the air, where t_A is the time it takes for v_y to become zero: $0 = v_{0Ay} - gt_A$.

A. A

[✓ B. B]

C. Both at the same time

$v_{0Ay} > v_{0By}$ because A goes higher.

Thus, $t_A > t_B$

The vertical part of the motion dictates the time a projectile spends in the air.

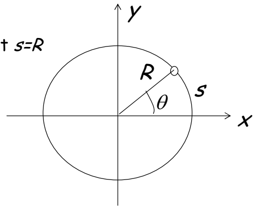
Circular Motion

Circular motion is the motion in a circle with constant radius.

Polar coordinates

Polar coordinates (r, θ) are more convenient than Cartesian coordinates to describe circular motion: $r = R$, only $\theta = \theta(t)$

Arch: $s = R\theta$
Definition: 1 radian = angle so that $s=R$
1 revolution = 2π radians



Relation to Cartesian coordinates:
$x = r\cos\theta$
$y = r\sin\theta$

Velocity

Cartesian coordinates: $v_x = \frac{dx}{dt}$; $v_y = \frac{dy}{dt}$

Polar coordinates: $\boxed{v_r = \frac{dr}{dt}}$; $\boxed{\omega = \frac{d\theta}{dt}}$

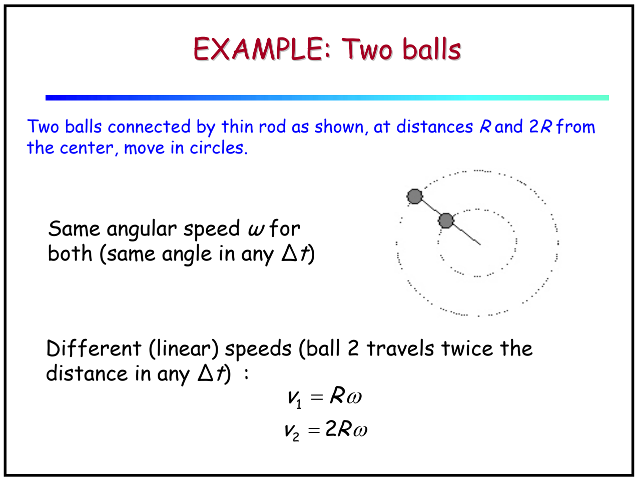
Radial velocity — Angular velocity

For circular motion:

$v_r = 0$

$$s = R\theta \longrightarrow \left|\frac{ds}{dt}\right| = R\left|\frac{d\theta}{dt}\right| \longrightarrow \boxed{v = R\omega}$$

(where ω is in radians/unit time)



EXAMPLE: Two balls

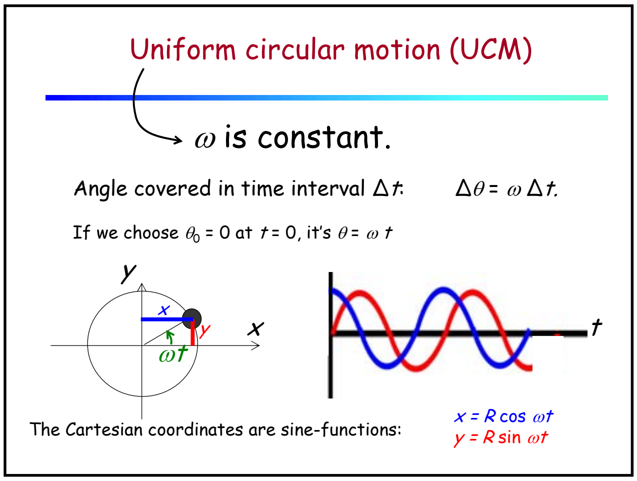
Two balls connected by thin rod as shown, at distances R and $2R$ from the center, move in circles.

Same angular speed ω for both (same angle in any Δt)

Different (linear) speeds (ball 2 travels twice the distance in any Δt) :

$$v_1 = R\omega$$

$$v_2 = 2R\omega$$



Uniform circular motion (UCM)

→ ω is constant.

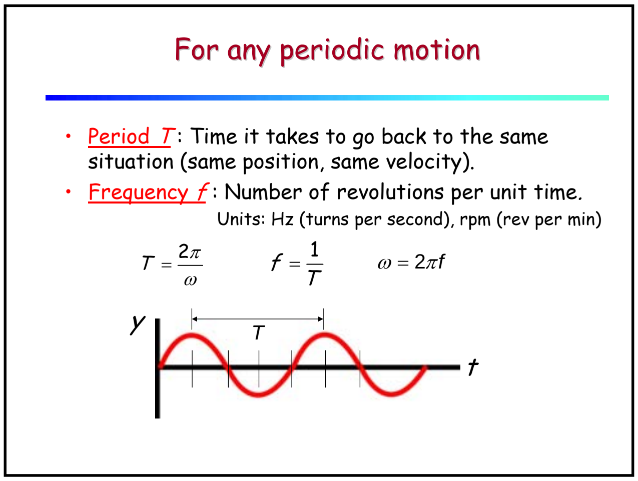
Angle covered in time interval Δt: $\Delta\theta = \omega\,\Delta t$.

If we choose $\theta_0 = 0$ at $t = 0$, it's $\theta = \omega\, t$

The Cartesian coordinates are sine-functions:

$x = R\cos\omega t$

$y = R\sin\omega t$

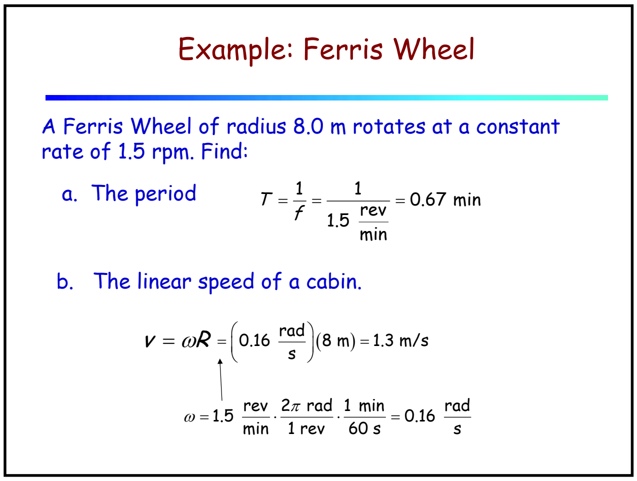


For any periodic motion

- Period T: Time it takes to go back to the same situation (same position, same velocity).
- Frequency f: Number of revolutions per unit time.

  Units: Hz (turns per second), rpm (rev per min)

$$T = \frac{2\pi}{\omega} \qquad f = \frac{1}{T} \qquad \omega = 2\pi f$$



Example: Ferris Wheel

A Ferris Wheel of radius 8.0 m rotates at a constant rate of 1.5 rpm. Find:

a. The period

$$T = \frac{1}{f} = \frac{1}{1.5\,\frac{\text{rev}}{\text{min}}} = 0.67\text{ min}$$

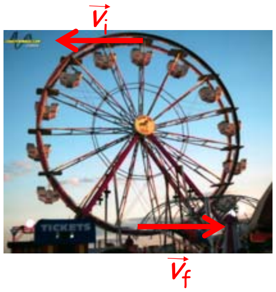
b. The linear speed of a cabin.

$$v = \omega R = \left(0.16\,\frac{\text{rad}}{\text{s}}\right)(8\text{ m}) = 1.3\text{ m/s}$$

$$\omega = 1.5\,\frac{\text{rev}}{\text{min}}\cdot\frac{2\pi\text{ rad}}{1\text{ rev}}\cdot\frac{1\text{ min}}{60\text{ s}} = 0.16\,\frac{\text{rad}}{\text{s}}$$

ACT: Ferris wheel

The ferris wheel in the figure rotates counterclockwise at a uniform rate. What is the direction of the average acceleration of a gondola as it goes from the top to the bottom of its trajectory?



A. Down

[✓ B. →]

C. The acceleration is 0 because the motion is uniform.

$$\vec{v}_f - \vec{v}_i \propto \vec{a}_{average}$$

Radial or centripetal acceleration

During uniform circular motion, speed is constant, but velocity is not!!! The direction keeps changing!

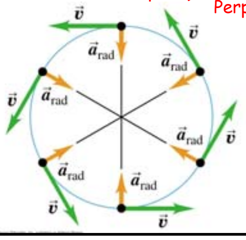
$$\vec{a} = \cancel{\vec{a}_{//}} + \boxed{\vec{a}_\perp}$$

(constant speed)

≠ 0

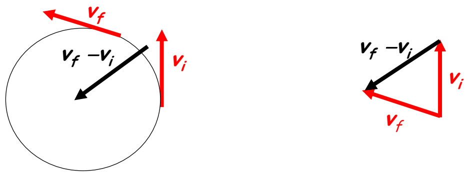
Perpendicular/normal/radial/centripetal acceleration

Points to the center of the circle



Radial acceleration

Visually: Let's look at the average acceleration.



$v_f - v_i$ points toward the center ⟶ a points toward the center

(Ok, it's sloppy: we should be taking the limit as $\Delta t \to 0$, but you get the idea…)

Magnitude of Radial Acceleration

For UCM:

$x = R \cos \omega t$
$y = R \sin \omega t$

$\xrightarrow{\frac{d}{dt}}$

$v_x = -R\,\omega \sin \omega t$
$v_y = R\,\omega \cos \omega t$

$\xrightarrow{\frac{d}{dt}}$

$a_x = -R\,\omega^2 \cos \omega t$
$a_y = -R\,\omega^2 \sin \omega t$

$|r| = R$ $\quad |v| = R\omega$ $\quad |a| = R\omega^2$

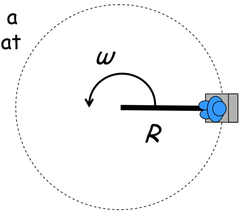
In UCM, all the acceleration is centripetal. Thus,

$$\boxed{a_r = R\omega^2 = \frac{v^2}{R}}$$

Radial acceleration

Acceleration simulator

In the movie "The Right Stuff", a seat at the end of a long arm that rotates very fast is used to prepare astronauts for high accelerations.



If $R = 5$ m, what is the speed needed to have $a = 5g$?

$$5g = R\omega^2$$

$$\omega = \sqrt{\frac{5g}{R}} \sim \sqrt{\frac{5\times10}{5}} = 3.2\ \frac{\text{rad}}{\text{s}}\times\frac{1\text{ turn}}{2\pi\text{ rad}} \sim 0.5\text{ turns/s}$$

Does this make sense? $|a_r| = \dfrac{v^2}{R}$

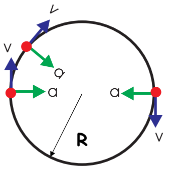
As you decrease the radius, the rate at which the velocity shifts, hence the acceleration, grows.

$$|a| \propto \frac{1}{R}$$

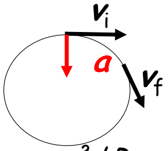
As you increase the velocity, the acceleration grows in two ways:

$$|a| \propto v^2$$

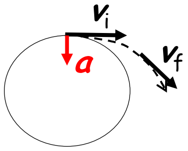
- The rate at which the velocity is shifting grows.
- The amount of velocity which needs to be shifted around also grows.



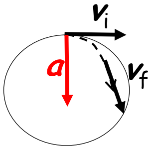
<u>The subtle point</u>: To have uniform circular motion, the acceleration needs to be **exactly** v^2/R.



$a = v^2/R$



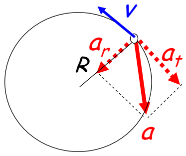
a too small; direction changes too little



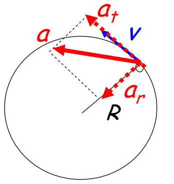
a too large; direction changes too much

Non-uniform circular motion

Slowing down



Speeding up

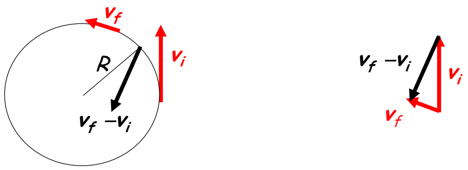


Radial acceleration (towards the center) → changes in direction

Tangential acceleration (tangential to trajectory) → changes in speed

Non-uniform circular motion

That sloppy but visual proof, for the slowing down case:



$\boldsymbol{v}_f - \boldsymbol{v}_i$ does not point to the center, so $\boldsymbol{a}$ does not point to the center either!

Angular acceleration

If linear speed v is changing, the angular speed ω is also changing

Angular acceleration $\boxed{\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}}$

$$v = R\omega \xrightarrow{\frac{d}{dt}} \boxed{a_t = R\alpha}$$

because $a_t = \frac{dv}{dt}$ (not $\vec{a} = \frac{d\vec{v}}{dt}$!!!)

Equations of motion when α = constant

The derivation is formally identical to what we did in 1D:

$$\begin{aligned} \omega &= \frac{d\theta}{dt} \\ \alpha &= \frac{d\omega}{dt} \end{aligned} \longrightarrow \boxed{\begin{aligned} &\omega = \omega_0 + \alpha t \\ &\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2 \\ \\ &\bar{\omega} = \frac{\Delta\theta}{\Delta t} = \frac{\omega + \omega_0}{2} \\ &\omega^2 - \omega_0^2 = 2\alpha\Delta\theta \end{aligned}}$$

Example: Grinding wheel

At t = 0, a grinding wheel has an angular velocity of 24.0 rad/s. It has an constant angular acceleration of 30.0 rad/s² until a circuit breaker trips at t = 2.00 s. From then on, it turns through 432 rad as it coasts to a stop at constant angular acceleration. What was its acceleration as it slowed down?

Part 1: Find angular speed at t = 2 s.

$$\omega = \omega_0 + \alpha t = 24.0 \text{ rad/s} + (30.0 \text{ rad/s}^2)(2.00 \text{ s})$$

$$\omega = 84 \text{ rad/s}$$

Part 2: Find angular acceleration.

$$\omega^2 - \omega_0^2 = 2\alpha\Delta\theta$$

$$\alpha = \frac{\omega^2 - \omega_0^2}{2\Delta\theta} = \frac{0 - (84 \text{ rad/s})^2}{2(432 \text{ rad})} = -8.17 \text{ rad/s}^2$$

Checks:

Sign ok

Units ok