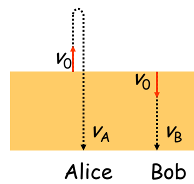
Lecture 5

2D and 3D motion

ACT: Alice and Bob

Alice and Bob stand at the top of a cliff of height h. Both throw a ball with initial speed v_0, Alice straight up and Bob straight down. The speed of the balls when they hit the ground are v_A and v_B, respectively. Which of the following is true?



A) $v_A < v_B$

B) $v_A = v_B$

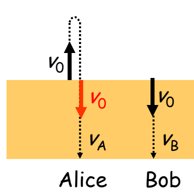
C) $v_A > v_B$

$y - y_0 = h$ ← $+v_0$ or $-v_0$

$v^2 = v_0^2 - 2gh$ same for both!

Most importantly: symmetry!

When the Alice's ball passes it initial position, its velocity is v_0 pointing down (just like Bob's)



$v_A = v_B$

2D (and 3D) motion

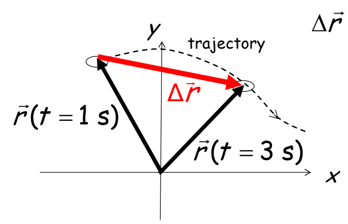
Now we need vectors to indicate position, velocity and acceleration, but the definitions we use in 1D are pretty much the same.

Position: $\vec{r}(t)$

Displacement:

$$\Delta\vec{r} = \vec{r}(t+\Delta t) - \vec{r}(t)$$

$$\left(\text{or } \Delta\vec{r} = \vec{r}_{final} - \vec{r}_{initial}\right)$$

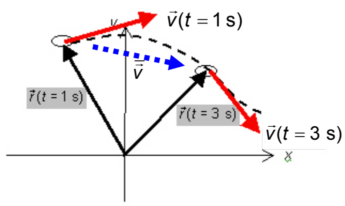


Velocity

Average: $\bar{\vec{v}} = \frac{\Delta \vec{r}}{\Delta t}$

Instantaneous: $\vec{v} = \frac{d\vec{r}}{dt}$ $\left(v_x = \frac{dx}{dt}, v_y = \frac{dy}{dt}, v_z = \frac{dz}{dt} \right)$

$\vec{v}$ is always tangent to the trajectory.



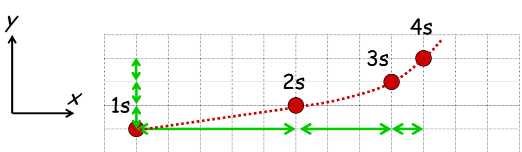
Acceleration

- Average: $\bar{\vec{a}} = \frac{\Delta \vec{v}}{\Delta t}$
- Instantaneous:

$$\vec{a} = \frac{d\vec{v}}{dt} \quad \left(a_x = \frac{dv_x}{dt}, a_y = \frac{dv_y}{dt}, a_z = \frac{dv_z}{dt} \right)$$

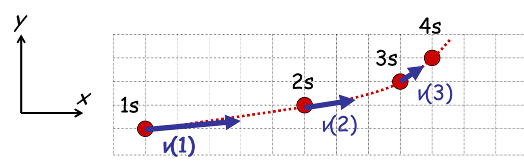
ACT: Acceleration

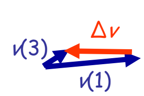
Shown below are the trajectory of a moving object and the snapshots taken every second. Which of the following is true about the components of the acceleration?



A) $a_x = 0$, $a_y > 0$ B) $a_x > 0$, $a_y > 0$ C) $a_x < 0$, $a_y = 0$

Note: Both the speed and the direction of velocity are changing!





The big new thing in 2D: changes in direction

Change in speed; parallel to v

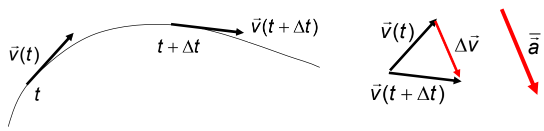
Change in direction; perpendicular to v

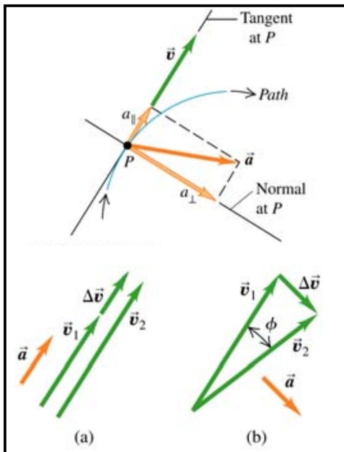
$$\vec{v} = v\hat{v} \qquad \vec{a} = \frac{d\vec{v}}{dt} = \frac{d(v\hat{v})}{dt} = \frac{dv}{dt}\hat{v} + v\frac{d\hat{v}}{dt}$$

An object can move at constant speed and still have $a \neq 0$!

This didn't happen in 1D!!

Graphically: Imagine an object moving along the following trajectory at constant speed. Take the positions at times t and Δt and find the average acceleration between them:





In 2 (or 3) dimensions, acceleration can occur both parallel to velocity or perpendicular to it

Acceleration in the direction of the velocity changes the speed.

Acceleration perpendicular to the velocity does not change the speed but shifts the direction of the motion.

Constant acceleration

- Same equations as in 1D but vectorial.

$$\vec{v} = \vec{v}_0 + \vec{a}t$$

$$\vec{r} = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2}\vec{a}t^2$$

(3 equations each)

$$\bar{\vec{v}} = \tfrac{1}{2}(\vec{v}_0 + \vec{v})$$

$$v^2 - v_0^2 = 2\vec{a}\cdot\Delta\vec{r}$$

(This is now a dot-product)

Important note: x, y and z are totally independent:

$v_x = v_{0x} + a_x t$	$v_y = v_{0y} + a_y t$	$v_z = v_{0z} + a_z t$
$x = x_0 + v_{0x}t + \frac{1}{2}a_x t^2$	$y = y_0 + v_{0y}t + \frac{1}{2}a_y t^2$	$z = z_0 + v_{0z}t + \frac{1}{2}a_z t^2$

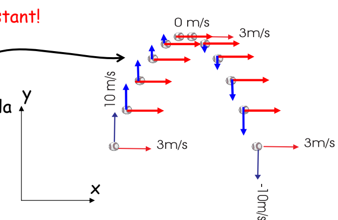
Projectiles

This an important example of a 2D problem.

We will consider projectiles in free fall where we neglect air resistance and curvature of the Earth.
With the most common choice of coordinate system:

- $a_y = -g$
- $a_x = 0$, v_x = constant!

The shape of the trajectory is a parabola



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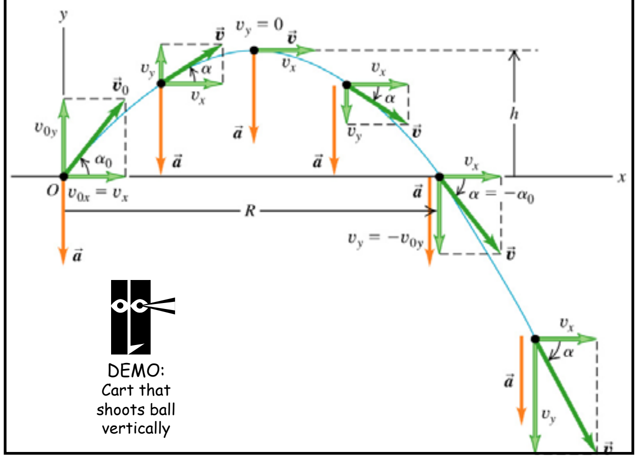
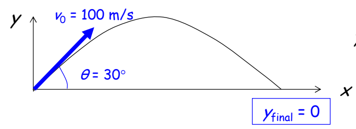


DEMO:
Cart that shoots ball vertically

Example: Projectiles

A projectile is fired from a cannon at a 30-degree angle with the ground and an initial velocity of 100 m/s. Assuming no air resistance and g = 10 m/s², calculate the time it will spend in the air.

a. 2.5 s b. 5.0 s c. 10 s d. 20 s e. 40 s



$$y = y_0 + v_{0y}t + \frac{1}{2}a_y t^2$$

$$0 = 0 + v_0 \sin\theta t - \frac{1}{2}gt^2 \longrightarrow \begin{cases} t = 0 \quad \text{(start!)} \\ t = \frac{2v_0}{g}\sin\theta = \frac{2(100\text{ m/s})}{10\text{ m/s}^2}\sin 30° = 10\text{ s} \end{cases}$$

ACT: Shoot the monkey

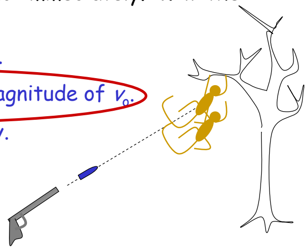
(tranquilizer gun)

A zookeeper shoots a tranquilizer dart to a monkey that hangs from a tree. He aims at the monkey and shoots a dart with an initial speed v_0. The monkey, startled by the gun, lets go immediately. Will the dart hit the monkey?

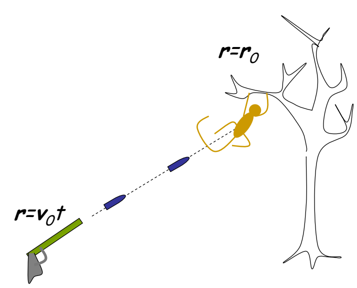
A) Only if v_0 is large enough.

B) Yes, regardless of the magnitude of v_0.

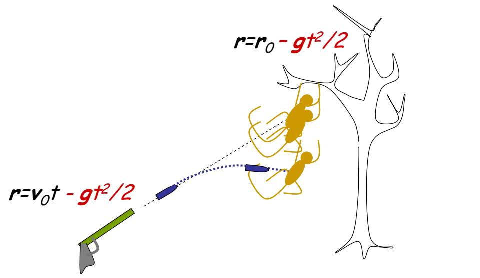
C) No, it misses the monkey.



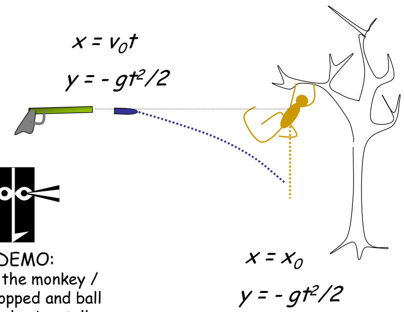
If there is no gravity, the dart hits the monkey…



If there is gravity, the dart also hits the monkey!



This might be easier to think about…



DEMO:
Shoot the monkey /
Ball dropped and ball
thrown horizontally

EXAMPLE: Mark McGwire.

Mark McGwire clobbers a fastball toward center-field. The ball is hit 1 m above the plate, and its initial velocity is 36.5 m/s at an angle of 30° above horizontal. The center-field wall is 113 m from the plate and is 3 m high.

a. How long after the hit does the ball reach the fence?

Mark McGwire clobbers a fastball toward center-field. The ball is hit 1 m above the plate, and its initial velocity is 36.5 m/s at an angle of 30° above horizontal. The center-field wall is 113 m from the plate and is 3 m high.

a. How long after the hit does the ball reach the fence?

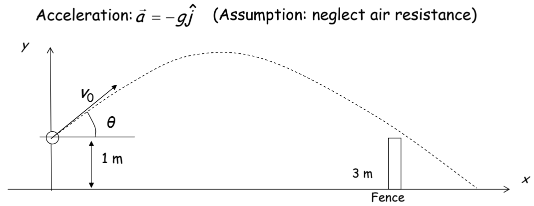
Draw a figure and select axes.

Identify initial and final situation:

Initial: $x_0 = 0$, $y_0 = 1$ m, $v_0 = 36.5$ m/s, $\theta = 30^\circ$

Final: $x = 113$ m (= "reaching the fence"), t? target

Acceleration: $\vec{a} = -g\hat{j}$ (Assumption: neglect air resistance)



Mark McGwire clobbers a fastball toward center-field. The ball is hit 1 m above the plate, and its initial velocity is 36.5 m/s at an angle of 30° above horizontal. The center-field wall is 113 m from the plate and is 3 m high.

We need an equation that relates final position to time.

$$x = x_0 + v_{0x} t + \frac{1}{2} a_x t^2 \quad \boxed{\text{but } a_x = 0}$$

$$x = 0 + v_{0x} t$$

$$v_{0x} = v_0 \cos\theta$$

$$t = \frac{x}{v_0 \cos\theta} = \frac{113\text{ m}}{(36.5\text{ m/s})(\cos 30^\circ)} = 3.58\text{ s}$$

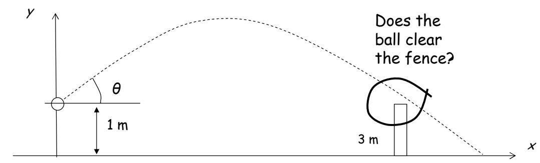
Checks:
Units ok
Reasonable ok
Limit $\theta = 90°$ ok

It takes 3.58s to reach the fence… assuming that it has not collided with the ground. To fully resolve this issue we move to part B.

Mark McGwire clobbers a fastball toward center-field. The ball is hit 1 m above the plate, and its initial velocity is 36.5 m/s at an angle of 30° above horizontal. The center-field wall is 113 m from the plate and is 3 m high.

b. Does Mark get a home run?

Does the ball go over the fence? → When x = 113 m, is y > 3m?



Mark McGwire clobbers a fastball toward center-field. The ball is hit 1 m above the plate, and its initial velocity is 36.5 m/s at an angle of 30° above horizontal. The center-field wall is 113 m from the plate and is 3 m high.

Let's find the value of y at t = 3.58 s
We need a relation which describes the evolution of y as a function of time:

$$y = y_0 + v_{0y} t + \frac{1}{2} a_y t^2 \xrightarrow[v_{0y} = v_0 \sin\theta]{} y = y_0 + v_0 \sin\theta\ t - \frac{1}{2} g t^2$$

$$y = 1.0\text{ m} + (36.5\text{ m/s})(\sin 30^\circ)(3.58\text{ s}) + \frac{1}{2}(-9.81\text{ m/s}^2)(3.58\text{ s})^2$$

$$= 3.5\text{ m} > 3\text{ m, homerun}$$

Checks: Units ok
Reasonable ok

Mark McGwire clobbers a fastball toward center-field. The ball is hit 1 m above the plate, and its initial velocity is 36.5 m/s at an angle of 30° above horizontal. The center-field wall is 113 m from the plate and is 3 m high.

c. What is the speed of the ball as it hits the ground?

"Hit the ground" : $y = 0$

To find v, we need a relation between y and the final velocity:

$$v^2 - v_0^2 = 2\vec{a} \cdot \Delta\vec{r}$$

$$v = \sqrt{v_0^2 + 2\vec{a} \cdot \Delta\vec{r}}$$

$$= \sqrt{v_0^2 - 2g\Delta y} \leftarrow \text{Only } y \text{ part of displacement!}$$

$$= \sqrt{(36.5\text{ m/s})^2 + 2(-9.81\text{ m/s}^2)(-1\text{ m})} = 36.8\text{ m/s}$$

Mark McGwire clobbers a fastball toward center-field. The ball is hit 1 m above the plate, and its initial velocity is 36.5 m/s at an angle of 30° above horizontal. The center-field wall is 113 m from the plate and is 3 m high.

An alternative (and waaaaay longer) approach:

$y = 0: \qquad 0 = y_0 + v_{0y}t - \frac{1}{2}gt^2$

$$0 = 1 + (36.5 \text{ m/s})\sin 30° t - \frac{1}{2}(9.81 \text{ m/s}^2)t^2 \qquad \rightarrow \qquad t = 3.77 \text{ s}$$

$$v_x = v_{0x} = v_0 \cos\theta = (36.5 \text{ m/s})\cos 30° = 31.6 \text{ m/s}$$

$$v_y = v_0 \sin\theta - gt = (36.5 \text{ m/s})\sin 30° - (9.81 \text{ m/s}^2)(3.77 \text{ s}) = -18.8 \text{ m/s}$$

$$v = \sqrt{v_x^2 + v_y^2} = 36.8 \text{ m/s}$$