Lecture 2

Vectors.

What is a vector?

- A scalar quantity is one that is represented by a single number. (Eg: Mass, length, time, temperature, volume.)
- A vector is a quantity which has both magnitude and direction. (Eg: Displacement, velocity, force.)

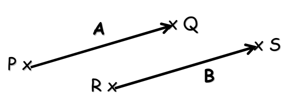
Notation: $\boldsymbol{A}$ or $\vec{A}$ (or $\underline{A}$)

Magnitude (how long): $|\boldsymbol{A}|$ or A

Direction: something like "Makes an angle of 36° with the horizontal as measured CCW"

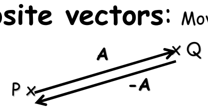
Vector basics

- **Equal vectors:** Moving from P to Q, and from R to S:



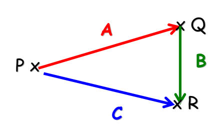
$\boldsymbol{A} = \boldsymbol{B}$

- **Opposite vectors:** Moving from P to Q, and from Q to P.



- **Unit vector ($\hat{A}$):** magnitude equals one. $\boldsymbol{A} = A\,\hat{A}$

Vector addition: Moving from P, to Q, to R

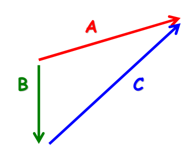


$\boldsymbol{C} = \boldsymbol{A} + \boldsymbol{B}$

Vector subtraction: It's an addition!

$\boldsymbol{A} - \boldsymbol{B} = \boldsymbol{A} + (-\boldsymbol{B})$

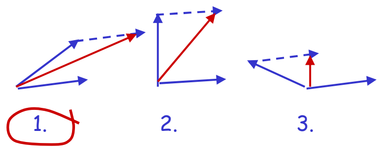
Visually: What do I have to add to B to get A?



$\boldsymbol{C} = \boldsymbol{A} - \boldsymbol{B}$

ACT: Vector addition

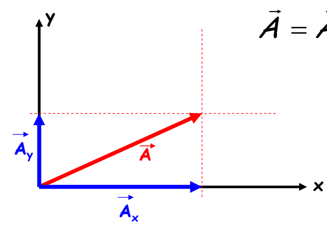
All the vectors below have the same magnitude. Which of the following arrangements will produce the largest resultant when the two vectors are added?



Components

A vector can be thought of as the vector-sum of the projections along the coordinate axes.

$$\vec{A} = \vec{A}_x + \vec{A}_y \left(+\vec{A}_z\right)$$



Cartesian unit vectors and components

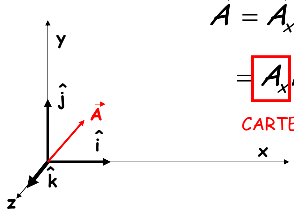
Some special unit vectors: $\hat{i}, \hat{j}, \hat{k}$

Any vector can be written in terms of these basic unit vectors:

$$\vec{A} = \vec{A}_x + \vec{A}_y + \vec{A}_z$$

$$= \boxed{A_x}\,\hat{i} + \boxed{A_y}\,\hat{j} + \boxed{A_z}\,\hat{k}$$

CARTESIAN COMPONENTS

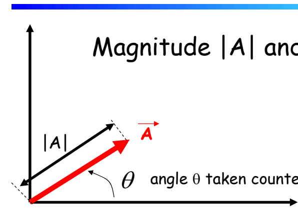


Another notation:

$A=(A_x,A_y,A_z)$

Polar Components (only 2D)

Magnitude |A| and direction θ



angle θ taken counterclockwise from the x axis

$$A_x = |A|\cos\theta$$

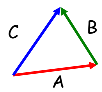
$$A_y = |A|\sin\theta$$

$$|A| = \sqrt{A_x^2 + A_y^2}$$

$$\theta = \arctan\left(\frac{A_y}{A_x}\right)$$

Vector addition in terms of components

- Geometric:



$\vec{A}+\vec{B}=\vec{C}$

- Algebraic: $A_x+B_x=C_x$

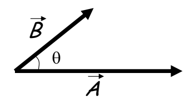
  $A_y+B_y=C_y$

  $A_z+B_z=C_z$

  i.e., do all the walking in the x-direction first, then all the walking in the y-direction, etc.

Scalar (dot) product

$$\vec{A}\cdot\vec{B}=|A||B|\cos\theta$$



Noteworthy

Parallel vectors

$$\vec{A}\cdot\vec{B}=AB$$

Antiparallel vectors

$$\vec{A}\cdot\vec{B}=-AB$$

Magnitude of a vector

$$|A|^2=\vec{A}\cdot\vec{A}$$

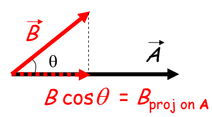
Perpendicular vectors

$$\vec{A}\cdot\vec{B}=0$$

Noteworthy

$$\vec{A}\cdot\vec{B}=AB_{\text{projected in the direction of A}}$$

$$=BA_{\text{projected in the direction of B}}$$



$$\vec{A}\cdot\vec{B}=A(B\cos\theta)=AB_{\text{projected in the direction of A}}$$

The dot product selects the part of ***B*** that is in the direction of ***A***.

Dot product in Cartesian components

$$\vec{A}\cdot\vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})\cdot(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}) =$$

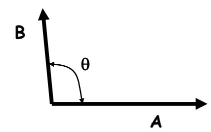
$$= A_xB_x\underbrace{(\hat{i}\cdot\hat{i})}_{1} + A_yB_x\underbrace{(\hat{j}\cdot\hat{i})}_{0} + \dots$$

$$\boxed{\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z}$$

Angle Between Vectors

We can use the dot product to determine the angle between two vectors:

$$\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta \qquad \boxed{\cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}||\vec{B}|}}$$



Example

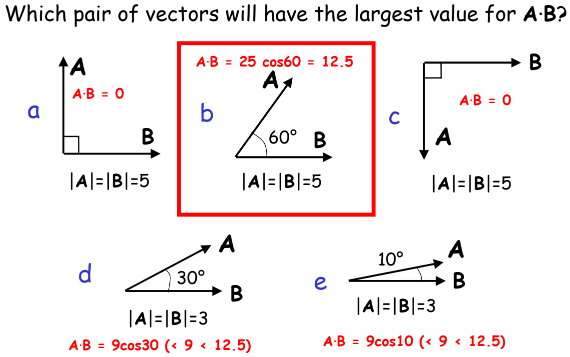
Find the angle between ***A*** = (2,-5,0) and ***B*** = (4,-1,3)

$$\cos\theta = \frac{\vec{A}\cdot\vec{B}}{|A||B|} = \frac{2\cdot 4 + (-5)(-1) + 3\cdot 0}{\sqrt{2^2+(-5)^2+0^2}\sqrt{4^2+(-1)^2+3^2}} = \frac{13}{\sqrt{754}}$$

$$\theta = \cos^{-1}\left(\frac{13}{\sqrt{754}}\right) = 61.7^\circ$$

Example: Dot product

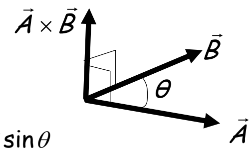
Which pair of vectors will have the largest value for ***A·B***?



Vector (or cross) product

The vector or cross product between two vectors

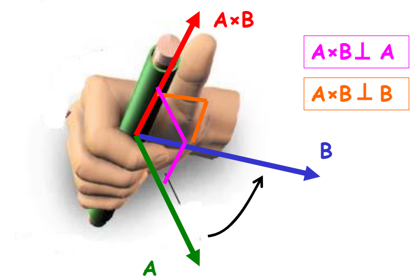
$$\vec{A} \times \vec{B}$$

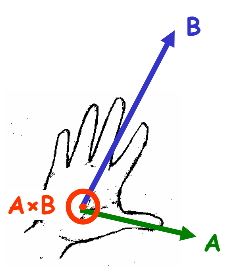


is a vector with:

- Magnitude: $\left|\vec{A} \times \vec{B}\right| = AB\sin\theta$
- Direction: Perpendicular to both A and B and as given by the right-hand rule.

Right-hand rule





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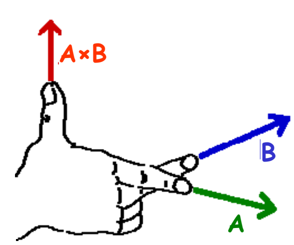

Noteworthy

- $\vec{A}\times\vec{B}$ is a vector! ($\vec{A}\cdot\vec{B}$ is a scalar)
- $\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$
- $\vec{A}\times\vec{A} = 0$
- $\hat{i}\times\hat{i} = 0, \quad \hat{j}\times\hat{j} = 0, \quad \hat{k}\times\hat{k} = 0$

  $\hat{i}\times\hat{j} = \hat{k}, \quad \hat{j}\times\hat{k} = \hat{i}, \quad \hat{k}\times\hat{i} = \hat{j}$

Cross product in cartesian components:

$$\vec{A}\times\vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})\times(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}) =$$

$$= A_xB_x\underbrace{(\hat{i}\times\hat{i})}_{0} + A_xB_y\underbrace{(\hat{i}\times\hat{j})}_{\hat{k}} + \ldots =$$

$$\vec{A}\times\vec{B} = (A_yB_z - A_zB_y)\hat{i} + (A_zB_x - A_xB_z)\hat{j} + (A_xB_y - A_yB_x)\hat{k}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$